

ERGODICITY OF AN SPDE ASSOCIATED WITH A MANY-SERVER QUEUE

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We introduce a two-component infinite-dimensional Markov process that serves as a diffusion model for a certain parallel server queue called the GI/GI/N queue, in the so-called Halfin-Whitt asymptotic regime. Under suitable assumptions on the service distribution, we characterize this process as the unique solution to a pair of stochastic evolution equations comprised of a real-valued Itô equation and a stochastic partial differential equation on the positive half line, which are coupled together by a nonlinear boundary condition. We construct an asymptotic (equivalent) coupling to show that this Markov process has a unique invariant distribution. These results are used in a companion paper [3] to show that the sequence of suitably scaled and centered stationary distributions of the GI/GI/N queue converges to the unique invariant distribution of the diffusion model, thus resolving (for a large class service distributions) an open problem raised by Halfin and Whitt in [23]. The methods introduced here are more generally applicable for the analysis of a broader class of networks.

CONTENTS

1	Introduction	1
2	Assumptions and Main Results	6
3	An Explicit Solution to the SPDE	12
4	Uniqueness of the Invariant Distribution	25
5	Proofs of Preliminary Results	36
A	Properties of the Renewal Equation	45
B	Properties of $\mathbb{H}^1(0, \infty)$	46
C	A Continuous Version of the Asymptotic Coupling Theorem	47
D	Verification of Assumptions for Certain Families of Distributions	48
	References	52
	Author's addresses	53

1. Introduction.

1.1. *Overview.* Many stochastic networks are typically too complex to be amenable to exact analysis. Instead, a common approach is to develop approximations that can be rigorously justified via limit theorems in a suitable asymptotic regime. Diffusion limits, which capture fluctuations of the state of the network around its mean behavior, and their invariant distributions have been well

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studied for networks of single server queues in heavy traffic (i.e., that are heavily loaded) when jobs in each class are served using a head-of-the-line scheduling policy. These limits are typically finite-dimensional diffusion processes (for example, reflected Brownian motions). On the other hand, when jobs in each class are served by a non-head-of-the-line scheduling policy such as earliest-deadline-first, processor sharing or shortest-remaining-processing-time, and the service distribution is general, natural Markovian state descriptors of the network are typically infinite-dimensional. Nevertheless, in heavy traffic, a sort of “state space collapse” typically occurs, which allows the infinite-dimensional diffusion limit to be expressed as a deterministic mapping of a one-dimensional stochastic process (see, e.g., [13, 33, 32, 19, 20]). In contrast, functional central limit theorems for many-server networks with general service distributions lead naturally to diffusion limits that are truly infinite-dimensional, and therefore require new techniques for their analysis. Whereas several limit theorems for many-server queues have been established (see Section 1.2 for a review) not much work has been devoted to the analysis of the associated limit processes.

The goal of this paper is to introduce some useful representations and techniques for the analysis of diffusion limits of many-server queues and their invariant distributions. The first contribution of this paper is to introduce a two-component Markov process (X, Z) that describes the asymptotic “diffusion-scaled” fluctuations of the state of a certain parallel-server queue called the GI/GI/N queue, in the so-called Halfin-Whitt asymptotic regime. The process (X, Z) , which we refer to as the diffusion model and describe precisely in Definition 3.9, takes values in a closed subset of $\mathbb{R} \times \mathbb{H}^1(0, \infty)$, where $\mathbb{H}^1(0, \infty)$ is the Hilbert space of square integrable functions on $(0, \infty)$ that have a square integrable weak derivative. The first component X represents the limit of the sequence $\{\hat{X}^N\}$, where \hat{X}^N is a centered, scaled version of the total number of jobs in the GI/GI/N system. The second component $Z = \{Z(t, \cdot), t \geq 0\}$ keeps track of just enough additional information so that (X, Z) is a Markov process. As elaborated in Section 2.2.1, (X, Z) can be viewed as a reduced version of the more complicated limit process obtained in [29] that is more amenable to analysis. Under suitable conditions on the service distribution, we characterize the components X and Z to be the unique solution of a coupled pair of stochastic equations driven by a Brownian motion and an independent space-time white noise (see Theorem 2.7). Specifically, X satisfies an Itô equation with a constant diffusion coefficient and a Z -dependent drift, and Z is an $\mathbb{H}^1(0, \infty)$ -valued process driven by a space-time white noise that satisfies a (non-standard) stochastic partial differential equation (SPDE) on the domain $(0, \infty)$. The two components are further linked by a nonlinear boundary condition: at each time $t \geq 0$, $Z(t, 0)$ is a (deterministic) nonlinear function of $X(t)$. A precise formulation of these equations is given in Definition 2.6. The SPDE characterization of (X, Z) facilitates the use of tools from stochastic calculus to compute performance measures of interest and natural generalizations would potentially also be useful for studying diffusion control problems for many-server networks, in a manner analogous to Brownian control problems that have been studied in the context of single-server networks [24].

Our second contribution (see Theorem 2.8) is to establish uniqueness of the invariant distribution of the Markov process (X, Z) . Standard methods for establishing ergodicity of finite-dimensional Markov processes such as positive Harris recurrence are not well suited to this setting due to the infinite-dimensional nature of the state space. Other techniques such as the dissipativity method used for studying ergodicity of nonlinear SPDEs (see, e.g. [11, 36]) also appear not immediately applicable due to the non-standard form of the equations, in particular, the presence of the non-linear boundary condition. Instead, we adopt the asymptotic (equivalent) coupling approach developed in a series of papers by Hairer, Mattingly, Scheutzow and co-authors (see, e.g., [14, 21, 5, 22] and

references therein) to show that (X, Z) has a unique invariant distribution. The asymptotic equivalent coupling that we construct has a somewhat different flavor from that used in previous works, and entails the analysis of a certain renewal equation. We believe that SPDEs with this kind of boundary condition are likely to arise in the study of scaling limits and control of other parallel server networks, and so our constructions could be useful in a broader context.

In a companion paper [3] (see also Section 2.2.1) we introduce an $\mathbb{H}^1(0, \infty)$ -valued process $Z^{(N)}$ which, together with $X^{(N)}$, serves as a state descriptor of the GI/GI/N queue, and show that the finite-dimensional projections of the centered and scaled sequence $\{(\hat{X}(t), \hat{Z}(t))\}$ converge in distribution to those of the corresponding marginals $Y(t) = (X(t), Z(t))$ of the diffusion model Y . Moreover, we also show in [3, Theorem 2.1 and Theorem 2.2] that in the Halfin-Whitt asymptotic regime, the corresponding sequence of stationary distributions of $(\hat{X}^{(N)}, \hat{Z}^{(N)})$ exists and converges to the unique invariant distribution of (X, Z) identified here. This resolves in the affirmative (for a large class of service distributions) the question of convergence of the scaled and centered steady-state distributions of GI/GI/N queues in the Halfin-Whitt regime posed in [23] and, reiterated in [16]. Furthermore, this allows one in principle to use the framework of Markov processes, although in an infinite-dimensional setting, to try to obtain a convenient characterization of the limit invariant distribution, or develop numerical schemes for its computation. This is relegated to future work. Finally, it is worth pointing out that the choice of the state space of (X, Z) is somewhat subtle (see Remark 2.10 for an elaboration of this point).

Reduced representations analogous to the one used here will likely be useful more generally for the asymptotic analysis of a large class of many-server networks with general service distributions. Indeed, a similar state representation has proved useful in [2, 1] for the asymptotic analysis of another many-server network model. In addition, this work also serves to illustrate the usefulness of the technique of asymptotic (equivalent) coupling for the study of stability properties of many-server stochastic networks. In Section 1.2, we review prior work in more detail, with an emphasis on the analysis in [28, 29], which is most closely related to this work.

1.2. Prior Work. The GI/GI/N queue consists of a network of N parallel servers to which a common stream of jobs arrive according to a renewal process and are processed in a First-Come-First-Serve manner (for a more detailed description, see [4, 23, 3] and Section 2.2.1). When the system is stable, an important performance measure is the steady state distribution of $X^{(N)}$, the total number of jobs in system, which includes those waiting in queue and those in service. A quantity of particular interest is the steady state probability that the queue is non-empty. An exact computation of this quantity is in general not feasible for large systems. However, when the service distribution G is exponential and the traffic intensity (which is the ratio of the average arrival rate to the average service rate) of the system has the form $1 - \beta N^{-1/2} + o(N^{-1/2})$ for some $\beta > 0$, and the interarrival distribution satisfies some minor technical conditions, Halfin and Whitt [23, Theorem 2] showed that the sequence of centered and renormalized processes $\hat{X}^{(N)} = (X^{(N)} - N)/\sqrt{N}$ converges weakly on finite time intervals to a positive recurrent diffusion X with a constant negative drift when $X > 0$ and an Ornstein-Uhlenbeck type restoring drift when $X < 0$. Moreover, they also showed [23, Proposition 1 and Corollary 2] that, as the number of servers N goes to infinity, the invariant distribution of $\hat{X}^{(N)}$ converges to the (unique) invariant distribution of the diffusion X . Since the exact form of this distribution can be easily identified, this provides a useful and explicit approximation for the steady state probability of an N -server queue being strictly positive for large N . Indeed, the asymptotic scaling for the traffic intensity mentioned above, which is commonly referred to as the Halfin-Whitt asymptotic regime, was chosen in [23] precisely to ensure that, in

the limit, the probability of a positive queue is non-trivial (i.e., lies strictly between zero and one), so as to provide a meaningful approximation for the corresponding quantity in the N -server system.

However, statistical analysis has shown that service distributions are typically non-exponential [9], and the problem of obtaining an analogous result for general, non-exponential service distributions was posed as an open problem in [23, Section 4]. This problem has remained unsolved except for a few specific distributions [17, 25, 46], even though tightness of the sequence of scaled queue-length processes was recently established under general assumptions by Gamarnik and Goldberg [16, 18]. The missing element in converting the tightness result of [16] to a convergence result was the identification and unique characterization of a candidate limit distribution. In analogy with the exponential case, a natural conjecture would be that the limit distribution is equal to the unique stationary distribution of the process X that is the limit (on every finite interval) of $\{\hat{X}^{(N)}\}$, the sequence of scaled processes, assuming that X can be shown to have a unique stationary distribution. However, whereas for exponential service distributions both the process $\hat{X}^{(N)}$ and its limit X are Markov processes, this is no longer true for more general service distributions. Indeed, although convergence (over finite time intervals) of the sequence of scaled processes $\{\hat{X}^{(N)}\}$ has been established for various classes of service distributions (see, e.g., [39, 35, 17, 40, 38, 29]), with the most general results obtained by Puhalskii and Reed in [40, 38], the obtained limit is not Markovian, with the exception of the work in [29]. This makes characterization of the stationary distribution of X challenging. It is easy to see that, except for special classes of distributions (e.g., phase-type distributions, as considered in [39]) any Markovian diffusion limit process will be infinite-dimensional. The key challenge is then to find a suitable diffusion model whose stationary distribution can be analyzed and shown to be the limit of the sequence of stationary distributions of a suitable diffusion-scaled state representation of the GI/GI/ N queue.

A Markovian state descriptor for the GI/GI/ N queue was proposed by Kaspi and Ramanan in [28, 29] in terms of a pair $(X^{(N)}, \nu^{(N)})$, where $\nu_t^{(N)}$ is a finite measure on $[0, \infty)$. A functional strong law of large numbers (or fluid) limit was established in [28] and the suitably renormalized fluctuations of the state descriptor around the scaled fluid limit (in the subcritical, critical and supercritical regimes) was shown to converge to a limit process (X, ν) in [29]. However, in the critical case, (X, ν) lies in a somewhat complicated space (the ν component is distribution-valued) and turns out not to be a Markov process on its own (i.e., the Markov property of the state representation is not preserved in the limit) unless one imposes very stringent assumptions on the service distribution, or one adds a third component to Markovianize the sight. A key insight that led to our simpler representation is the observation that the first and third components (when the latter is chosen to be on a suitable space) on their own form a nice Markov process.

For the simpler case of infinite-server queues, the Z process in our representation can be shown to be Markovian on its own, there is no nonlinear coupling with X to deal with, and the dynamics are simpler to describe (via a linear SPDE). In this case, it is more straightforward to establish uniqueness and identify the form of the invariant distribution without resorting to any asymptotic coupling argument. Moyal and Decreusefond [12], and subsequently Reed and Talreja [41], used a different representation of the state in the space of tempered distributions, and showed that the diffusion limit is a tempered distribution-valued Ornstein Uhlenbeck process. While this choice of state space facilitates the analysis, in particular allowing Reed and Talreja [41] to show that the invariant distribution of this diffusion limit is an explicit (infinite-dimensional) Gaussian distribution, it requires stronger conditions on the service distribution, such as infinite differentiability of the hazard rate function and boundedness of all its derivatives (see [Assumption 1.2][41]).

1.3. *Outline of the Rest of the Paper.* In Section 2, we introduce the assumptions on the service distribution, introduce the *diffusion model SPDE*, and state our main results, Theorem 2.7 and Theorem 2.8. In Section 3, we provide an explicit construction of the diffusion model (X, Z) and show that it is the unique solution to the diffusion model SPDE, and is also a time-homogeneous Feller Markov process. The proof of Theorem 2.7 is given at the end of Section 3.5. In Section 4, which is devoted to the proof of Theorem 2.8, we construct a suitable asymptotic equivalent coupling to show that the diffusion model has at most one invariant distribution. Various regularity properties of certain stochastic integrals and mappings that are stated in Section 3.1 and used in the proofs are established in Section 5. Proofs of a few additional results are relegated to Appendices B and A. First, in Section 1.4, we introduce common notation used throughout the paper.

1.4. *Some Common Notation.* The following notation will be used throughout the paper. \mathbb{Z} , \mathbb{Z}_+ and \mathbb{N} are the sets of integers, nonnegative integers and positive integers, respectively. Also, \mathbb{R} is set of real numbers and \mathbb{R}_+ the set of nonnegative real numbers. For $a, b \in \mathbb{R}$, $a \wedge b$ and $a \vee b$ denote the minimum and maximum of a and b , respectively. Also, $a^+ \doteq a \vee 0$ and $a^- \doteq -(a \wedge 0)$. For a set B , $\mathbb{1}_B(\cdot)$ is the indicator function of the set B (i.e., $\mathbb{1}_B(x) = 1$ if $x \in B$ and $\mathbb{1}_B(x) = 0$ otherwise). Moreover, with a slight abuse of notation, on every domain V , $\mathbf{1}$ denotes the constant function equal to 1 on V .

For every $n \in \mathbb{N}$ and subset $V \subset \mathbb{R}^n$, $\mathbb{C}(V)$, $\mathbb{C}_b(V)$ and $\mathbb{C}_c(V)$ are respectively, the space of continuous functions on V , the space of bounded continuous functions on V and the space of continuous functions with compact support on V . For $f \in \mathbb{C}[0, \infty)$ and $T > 0$, $\|f\|_T$ denotes the supremum of $|f(s)|$ over $s \in [0, T]$, and $\|f\|_\infty$ denotes the supremum of $|f(s)|$ over $[0, \infty)$. A function for which $\|f\|_T < \infty$ for every $T < \infty$ is said to be locally bounded. Also, $\mathbb{C}^0[0, \infty)$ denotes the subspace of functions $f \in \mathbb{C}[0, \infty)$ with $f(0) = 0$, $\mathbb{C}^1[0, \infty)$ denotes the set of functions $f \in \mathbb{C}[0, \infty)$ for which the derivative, denoted by f' , exists and is continuous on $[0, \infty)$ (with $f'(0)$ denoting the right derivative at 0), and $\mathbb{C}_b^1[0, \infty)$ represents the subset of functions in $\mathbb{C}^1[0, \infty)$ that are bounded and have a uniformly bounded derivative. Moreover, for every Polish space \mathcal{X} , $\mathbb{C}([0, \infty); \mathcal{X})$ denotes the set of continuous \mathcal{X} -valued functions on $[0, \infty)$. Recall that when $V \subset \mathbb{R}$, a function $f : V \mapsto \mathbb{R}$ is uniformly continuous on an interval $I \subset V$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $t, s \in I$ with $|t - s| < \delta$, $|f(t) - f(s)| \leq \epsilon$. A function $f : (0, \infty) \mapsto \mathbb{R}$ is called locally uniformly continuous if, for every $0 < T < \infty$, it is uniformly continuous on the interval $I = (0, T)$. Note that for a locally uniformly continuous function f , the limit $\lim_{t \downarrow 0} f(t)$ exists and f can be continuously extended to $[0, \infty)$ by setting $f(0) \doteq \lim_{t \downarrow 0} f(t)$.

Let $\mathbb{L}^1(0, \infty)$, $\mathbb{L}^2(0, \infty)$, and $\mathbb{L}^\infty(0, \infty)$, denote, respectively, the spaces of integrable, square-integrable and essentially bounded measurable functions on $(0, \infty)$, equipped with their corresponding standard norms. Also, let $\mathbb{L}_{\text{loc}}^1(0, \infty)$ denote the space of locally integrable functions on $[0, \infty)$. For any $f \in \mathbb{L}_{\text{loc}}^1(0, \infty)$ and a function g that is bounded on finite intervals, $g * f$ denotes the (one-sided) convolution of two functions, defined as $f * g(t) \doteq \int_0^t f(t - s)g(s)ds$, $t \geq 0$. Note that $f * g$ is locally integrable and locally bounded. Let $\mathbb{H}^1(0, \infty)$ denote the space of square integrable functions f on $(0, \infty)$ whose weak derivative f' exists and is also square integrable, equipped with the norm

$$\|f\|_{\mathbb{H}^1} = \left(\|f\|_{\mathbb{L}^2(0, \infty)}^2 + \|f'\|_{\mathbb{L}^2(0, \infty)}^2 \right)^{\frac{1}{2}}.$$

The space $\mathbb{H}^1(0, \infty)$ is a separable Banach space, and hence, a Polish space (see e.g. [8, Proposition 8.1 on page 203]). Also, for a function $t \mapsto \{u(t, r), r > 0\} \in \mathbb{C}([0, \infty), \mathbb{H}^1(0, \infty))$, $\partial_r u(t, \cdot)$ denotes the weak derivative of $u(t, \cdot)$ for every $t \geq 0$. Finally, recall that every function $f \in \mathbb{H}^1(0, \infty)$ is

almost everywhere equal to an absolutely continuous function whose derivative coincides with the weak derivative of f , almost everywhere [15, Problem 5 on p. 290].

Finally, for two measures μ, ν on a measurable space (Ω, \mathcal{F}) , μ is said to be absolutely continuous with respect to ν , denoted $\mu \ll \nu$, if for every subset $A \in \mathcal{F}$, $\nu(A) = 0$ implies $\mu(A) = 0$. When $\mu \ll \nu$ and $\nu \ll \mu$, μ and ν are said to be equivalent, and this is denoted by $\mu \sim \nu$.

2. Assumptions and Main Results.

2.1. *Assumptions.* Throughout G is a right-continuous non-decreasing function that satisfies $G(0) = 0$ and $G(x) \rightarrow 1$ as $x \rightarrow \infty$. The function G represents the cumulative distribution function (cdf) of the service distribution. We also use $\bar{G} \doteq 1 - G$ to denote the complementary cdf.

ASSUMPTION I. The function G satisfies the following properties:

- a. G is continuously differentiable with a derivative g and $\int_0^\infty \bar{G}(x)dx < \infty$.
- b. The function h defined by

$$(2.1) \quad h(x) \doteq \frac{g(x)}{\bar{G}(x)}, \quad x \in [0, \infty),$$

is uniformly bounded, that is, $H \doteq \sup_{x \in [0, \infty)} h(x) < \infty$.

- c. The derivative g is continuously differentiable with derivative g' , and the function

$$h_2(x) \doteq \frac{g'(x)}{\bar{G}(x)}, \quad x \in [0, \infty),$$

is uniformly bounded, that is, $H_2 \doteq \sup_{x \in [0, \infty)} |h_2(x)| < \infty$.

REMARK 2.1. Assumption I.a implies that the service distribution has a continuous probability density function (p.d.f.) g and finite mean. By changing units if necessary, we can (and will) assume without loss of generality that the service distribution has mean 1, that is, $\int_0^\infty G(x)dx = 1$. Note that h represents the hazard rate function of the service time distribution, and the boundedness of h implies that the support of the service time distribution is all of $[0, \infty)$.

ASSUMPTION II. For some $\epsilon > 0$, $\bar{G}(x) = \mathcal{O}(x^{-2-\epsilon})$ as $x \rightarrow \infty$, that is, there exists a finite positive constant c such that $\bar{G}(x) \leq c x^{-2-\epsilon}$ for all sufficiently large x . In other words, the service time distribution has a finite $(2 + \epsilon)$ moment.

REMARK 2.2. Since $\bar{G}(x) \leq 1$ and $\int_0^\infty \bar{G}(x)dx = 1$ by Remark 2.1, \bar{G} lies in $\mathbb{L}^1(0, \infty) \cap \mathbb{L}^2(0, \infty)$. Hence, Assumptions I.b and I.c, respectively, imply that g and g' also lie in $\mathbb{L}^1(0, \infty) \cap \mathbb{L}^2(0, \infty)$. Also, Assumption II along with the fact that $\int_0^\infty \bar{G}(x)dx \leq 1$ implies that $\int_0^\infty \bar{G}(x)dx$ also lies in $\mathbb{L}^1(0, \infty) \cap \mathbb{L}^2(0, \infty)$.

It is easily verified (see Appendix D) that Assumptions I and II are satisfied by a large class of distributions of interest, including phase-type distributions, Gamma distributions with shape parameter $\alpha \geq 2$, Lomax distributions (i.e., generalized Pareto distributions with location parameter $\mu = 0$) with shape parameter $\alpha > 2$, and the log-normal distribution, which has been empirically observed to be a good fit for service distributions arising in applications [9, Section 4.3].

2.2. The Diffusion Model SPDE. In Section 2.2.2 we introduce the an SPDE, which is shown in Theorem 2.7 to uniquely characterize the diffusion model $Y = (X, Z)$. To motivate the form of the SPDE, in Section 2.2.1 we first recall both the equations governing the dynamics of the GI/GI/N queue and of the (critical) limit process (X, ν) obtained in [29]. We then formally derive the diffusion model SPDE from these equations. It should be stressed that this is not a rigorous derivation, but only serves to provide intuition into the form of the SPDE. In particular, this derivation is not used in the rest of the paper and so can be skipped without loss of continuity.

We first introduce some notation common to Sections 2.2.1 and 2.2.2. The diffusion model SPDE is driven by a Brownian motion B and an independent space-time white noise on $[0, \infty)^2$ based on the measure $g(x)dx \otimes dt$, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In other words, $\mathcal{M} = \{\mathcal{M}_t(A), A \in \mathcal{B}[0, \infty), t \in [0, \infty)\}$ is a continuous martingale measure with covariance

$$(2.2) \quad \langle \mathcal{M}(A), \mathcal{M}(\tilde{A}) \rangle_t = t \int_0^\infty \mathbb{1}_{A \cap \tilde{A}}(x) g(x) dx, \quad A, \tilde{A} \in \mathcal{B}([0, \infty)).$$

For definitions and properties of martingale measures and space-time white noise, we refer the reader to [45, Chapters 1 and 2], and for the definition of the quadratic variation process $\langle M \rangle$ and cross variation process $\langle M, N \rangle$ of martingales M and N , see [27, Definition 5.3 and Definition 5.5 of Chapter 1]. For every bounded measurable function φ on $[0, \infty) \times [0, \infty)$, for $t \geq 0$, the stochastic integral of φ on $[0, \infty) \times [0, t]$ with respect to the martingale measure \mathcal{M} is denoted by

$$(2.3) \quad \mathcal{M}_t(\varphi) \doteq \iint_{[0, \infty) \times [0, t]} \varphi(x, s) \mathcal{M}(dx, ds).$$

Note that $\mathcal{M}_t(\varphi)$ is a continuous Gaussian process with independent increments. Also, let $\tilde{\mathcal{F}}_t \doteq \sigma(B(s), \mathcal{M}_s(A); 0 \leq s \leq t, A \in \mathcal{B}(0, \infty))$, and let the filtration $\{\mathcal{F}_t\}$ denote the augmentation (see [27, Definition 7.2 of Chapter 2]) of $\{\tilde{\mathcal{F}}_t\}$ with respect to \mathbb{P} . Recall that the stochastic integral $\{\mathcal{M}_t(\varphi), t \geq 0\}$ is an $\{\mathcal{F}_t\}$ -martingale with (predictable) quadratic variation

$$(2.4) \quad \langle \mathcal{M}(\varphi) \rangle_t = \int_0^t \int_0^\infty \varphi^2(x, s) g(x) dx ds.$$

As elaborated in Section 2.2.1, B and \mathcal{M} arise as limits of suitably scaled fluctuations of the arrival and departure processes, respectively, in the GI/GI/N model.

2.2.1. A Formal Derivation of the Diffusion Model SPDE. As mentioned in the introduction, in [29], a Markovian state descriptor $(X^{(N)}, \nu^{(N)})$ was introduced for the GI/GI/N queue, where $X^{(N)}(t)$ represents the total number of jobs in system at time t and $\nu_t^{(N)}$ is the finite measure that is the sum of unit Dirac masses at the ages (i.e., time spent in service) of jobs in service at time t . We now recall the equations governing the dynamics of $(X^{(N)}, \nu^{(N)})$, which were derived in [28, Theorem 5.1] (see also [26, Appendix A]). Let $E^{(N)}(t)$ and $D^{(N)}(t)$ be the cumulative number of jobs that respectively, arrived into and departed from the system in the interval $[0, t]$. Then, a simple mass balance for the number of jobs in system implies

$$(2.5) \quad X^{(N)}(t) = X^{(N)}(0) + E^{(N)}(t) - D^{(N)}(t).$$

Next, since $\nu_t^{(N)}(\mathbf{1})$ is the number of jobs in service (recall that $\mathbf{1}$ denotes the function identically equal to 1 and $\nu_t^{(N)}(\mathbf{1})$ is the integral of $\mathbf{1}$ with respect to $\nu_t^{(N)}$), the non-idling property implies

$$(2.6) \quad (X^{(N)}(t) - N) \wedge 0 = \nu_t^{(N)}(\mathbf{1}) - N.$$

Finally, let $Q_t^{(N)}$ be a finite measure equal to the sum of Dirac delta masses at the (total) service time requirements of all jobs that departed during $[0, t]$. Then the number of departures during $[0, t]$ is given by $D^{(N)}(t) = Q_t^{(N)}(\mathbf{1})$ ¹. The process $\nu^{(N)}$ satisfies the following equation: for every sufficiently smooth function f on $[0, \infty)$ with derivative f' ,

$$(2.7) \quad \nu_t^{(N)}(f) = \nu_0^{(N)}(f) + \int_0^t \nu_s^{(N)}(f') ds - Q_t^{(N)}(f) + f(0)K^{(N)}(t),$$

where $K^{(N)}(t)$ is the number of jobs that entered service during $[0, t]$, and satisfies the following mass balance equation [28, equation (2.6)]:

$$(2.8) \quad K^{(N)}(t) = \nu_t^{(N)}(\mathbf{1}) - \nu_0^{(N)}(\mathbf{1}) + D^{(N)}(t).$$

Under the assumption that the arrival process $E^{(N)}$ is a renewal process with rate $\lambda^{(N)}$, and the service distribution has finite mean and a density, it was shown in [28, Theorem 3.7] that $(X^{(N)}, \nu^{(N)})/N$ converges, as $N \rightarrow \infty$, to a continuous “fluid limit” $(\bar{X}, \bar{\nu})$, which is the unique solution to a system of coupled equations referred to as the fluid equations. It was also shown in [28, Remark 3.8] that the pair $(\bar{X}^*, \bar{\nu}^*(dx)) \doteq (1, \bar{G}(x)dx)$ is an invariant state for the fluid equations. A corresponding functional central limit theorem was established in [29]. Specifically, it was shown in [29, Remark 5.1] that under the Halfin-Whitt scaling, the centered and scaled renewal arrival process $\hat{E}^{(N)}(t) \doteq (E^{(N)}(t) - \lambda t)/\sqrt{N}; t \geq 0$, converges to a process E of the form $E(t) = \sigma B(t) - \beta t$, where B is a Brownian motion, β is the constant in the Halfin-Whitt scaling, and $\sigma > 0$ is a constant that depends on the mean and variance of the interarrival times [29, Remark 5.1]. It was also shown in [28, Corollary 5.5] that the compensator (with respect to a suitable filtration) of the process $Q^{(N)}(f)$ is given by $\int_0^\cdot \nu_s^{(N)}(hf) ds$, where h is the hazard rate function defined in (2.1) and in [28, Lemma 5.9] and [29, Corollary 8.3] that the sequence of scaled local martingales $\hat{\mathcal{M}}^{(N)}(f) = (Q^{(N)}(f) - \int_0^\cdot \nu_s^{(N)}(hf) ds)/\sqrt{N}$ converges to $\mathcal{M}(f)$, where \mathcal{M} is a martingale measure with $\langle \mathcal{M}(A), \mathcal{M}(\tilde{A}) \rangle_t = \int_0^t \bar{\nu}_s(h \mathbf{1}_{A \cap \tilde{A}}) ds$, and $\{\bar{\nu}_s, s \geq 0\}$ is the fluid limit. In the critical case and under the assumption that the sequence of fluid-scaled initial conditions converges to the invariant state $(\bar{X}^*, \bar{\nu}^*)$, since $\bar{\nu}_s(dx) \equiv \bar{G}(x)dx$, the covariance functional of \mathcal{M} takes the form given in (2.2). Furthermore, by the asymptotic independence result established in [29, Proposition 8.4], \mathcal{M} and B are independent.

Finally, it was shown in [29, Theorem 3] that the scaled and centered processes $(\hat{X}^{(N)}, \hat{\nu}^{(N)}) = (X^{(N)} - N\bar{X}^*, \nu^{(N)} - N\bar{\nu})/\sqrt{N}$ converge to a limit process (X, ν) , where X is a real-valued continuous stochastic process and ν takes values in a distribution space (specifically, \mathbb{H}_{-2} , although we do not provide the precise definition of this space here as it is not relevant for our informal discussion). It follows from [29, Theorem 5 and eqn. (5.28)] that ν satisfies the following equation:

$$(2.9) \quad \nu_t(f) = \nu_0(f) + \int_0^t \nu_s \left(\frac{df}{dx} - hf \right) ds - \mathcal{M}_t(f) + f(0)K(t),$$

for all sufficiently regular f . Here, $K(t)$ is the limit of the centered and scaled number of jobs that entered service in $[0, t]$, and satisfies (as can be deduced from Theorem 5 and equation (6.8) of [29]):

$$K(t) = \nu_t(\mathbf{1}) - \nu_0(\mathbf{1}) + \int_0^t \nu_s(h) ds + \mathcal{M}_t(\mathbf{1}).$$

¹ $Q_t^{(N)}(f)$ is denoted in [28] by $Q_f^{(N)}(t)$

The term $\int_0^t \nu_s(h) ds$ represents the fluctuations in the compensator of the departure process, and $\mathcal{M}_t(\mathbf{1})$ is the limit of the scaled compensated departure process, and thus, their sum captures the fluctuations in the total departures from the system. A drawback of the limit process (X, ν) is that it lies in the rather complicated state space $\mathbb{R} \times \mathbb{H}_{-2}(0, \infty)$. Moreover, the functions $\mathbf{1}$ and h do not in general lie in the dual space \mathbb{H}_2 of \mathbb{H}_{-2} , and thus, in [29], $\nu_t(\mathbf{1})$ and $\nu_t(h)$, which appear in the dynamic equations above, are not well-defined as functionals of ν_t , but instead represent limits of $\widehat{\nu}_t^{(N)}(\mathbf{1})$ and $\widehat{\nu}_t^{(N)}(h)$. Consequently, the limit process (X, ν) is not Markov.

The simpler diffusion model proposed in this paper is based on the key observation that, instead of considering the whole distribution-valued process ν , it suffices to only consider the action of ν on the one-parameter family of functions $\{\vartheta^r; r \geq 0\}$ defined as follows: for $r \geq 0$,

$$(2.10) \quad \vartheta^r(x) \doteq \frac{\overline{G}(x+r)}{\overline{G}(x)}, \quad x \geq 0.$$

Specifically, formally set $Z(t, r) = \nu_t(\vartheta^r)$, $t, r \geq 0$. Then $Z(t, r)$ admits an interpretation as the limit of the fluctuations of the prelimit quantity

$$Z^{(N)}(t, r) \doteq \nu_t^{(N)}(\vartheta^r) = \sum_j \frac{\overline{G}(a_j^{(N)}(t) + r)}{\overline{G}(a_j^{(N)}(t))},$$

where the summation is over the indices of jobs in service at time t . Here, $Z^{(N)}(t, r)$ is the conditional expected number of jobs that were in service at time t and still have not left the system by time $t+r$, given the ages of jobs in service up to time t . A type of convergence of the centered and scaled sequence $\{\widehat{Y}^{(N)}(t) = (\widehat{X}^{(N)}(t), \widehat{Z}^{(N)}(t))\}$, where $\widehat{Z}^{(N)}(t)$ captures the scaled fluctuations of $Z^{(N)}(t)$ around its fluid limit at time t , to $Y(t) = (X(t), Z(t))$ is established in [3, Proposition 7.2].

Substituting $f = \vartheta^r$ in (2.9), noting that $\partial_x \vartheta^r - h \vartheta^r = \partial_r \vartheta^r$ and $\vartheta^r(0) = \overline{G}(r)$, and (blithely) assuming we have enough regularity to justify the equality $\partial_r Z(s, r) = \nu_s(\partial_r \vartheta^r)$ for $r \geq 0$, which (since $\partial_r \vartheta^r|_{r=0} = -h$) in particular implies

$$(2.11) \quad \partial_r Z(s, 0) = -\nu_s(h),$$

we obtain

$$(2.12) \quad Z(t, r) = Z(0, r) + \int_0^t \partial_r Z(s, r) ds - \mathcal{M}_t(\vartheta^r) + \overline{G}(r)K(t),$$

where, observing that $\vartheta^0 = \mathbf{1}$ and so $Z(t, 0) = \nu_t(\mathbf{1})$, the equation for K can be rewritten as

$$(2.13) \quad K(t) = Z(t, 0) - Z(0, 0) - \int_0^t \partial_r Z(s, 0) ds + \mathcal{M}_t(\mathbf{1}).$$

Moreover, on substituting $\nu_s(h) = \partial_r Z(s, 0)$ from (2.11) in the equation for X obtained in [29, Corollary 5.9] and [29, Theorem 2 and eqn. (5.16)] we see that X satisfies the Itô equation

$$(2.14) \quad X(t) = X_0 + \sigma B(t) - \beta t - \mathcal{M}_t(\mathbf{1}) - \int_0^t \partial_r Z(s, 0) ds.$$

In addition, the N -server equation (2.6) transforms into the nonlinear boundary condition $X(t) \wedge 0 = \nu_t(\mathbf{1})$. Observing that $\vartheta^0 = \mathbf{1}$ and substituting $\nu_t(\mathbf{1}) = Z(t, 0)$, this can be rewritten as

$$(2.15) \quad X(t) \wedge 0 = Z(t, 0).$$

Thus, equations (2.12)–(2.15) suggest that the pair (X, Z) can serve as an alternative and much simpler diffusion model when compared to (X, ν) .

The informal derivation given above made several assumptions without justification, such as the interchange in (2.11) (which actually only needs to hold in integral form), and also does not properly address the regularity of $Z(t, r)$ with respect to r or, equivalently, the space in which $Z(t, \cdot)$ lies, which cannot be inferred from results in [29]. As elaborated in Remark 2.10, a suitable choice of the state space for Z is rather subtle. Indeed, $Z(t, \cdot)$, when viewed as a $\mathbb{C}[0, \infty)$ -valued process, was appended to the limit obtained in [29] and it was shown there that (X, ν, Z) is a time-inhomogeneous Feller Markov process. However, as shown below, when Z is viewed instead as an $\mathbb{H}^1(0, \infty)$ -valued process, the much simpler state representation (X, Z) yields on its own a time-homogeneous Feller Markov process.

2.2.2. The Diffusion Model SPDE. We first introduce the state space \mathbb{Y} of the diffusion model. Recall the following properties of the function space $\mathbb{H}^1(0, \infty)$.

LEMMA 2.3. *The function space $\mathbb{H}^1(0, \infty)$ satisfies the following properties:*

- a. *For every function $f \in \mathbb{H}^1(0, \infty)$, there exists a (unique) function $f^* \in \mathbb{C}[0, \infty)$ such that $f = f^*$ a.e. on $(0, \infty)$.*
- b. *The embedding $I : \mathbb{H}^1(0, \infty) \mapsto \mathbb{C}[0, \infty)$ that takes f to f^* is continuous.*
- c. *For every $t \geq 0$, the mapping $f \mapsto f(t + \cdot)$ is a continuous mapping from $\mathbb{H}^1(0, \infty)$ into itself. Also, for every $f \in \mathbb{H}^1(0, \infty)$, the translation mapping $t \mapsto f(t + \cdot)$ from $[0, \infty)$ to $\mathbb{H}^1(0, \infty)$ is continuous. Moreover,*

$$(2.16) \quad \lim_{t \rightarrow \infty} \|f(t + \cdot)\|_{\mathbb{H}^1} = 0.$$

PROOF. Part a is proved in [8, Theorem 8.2]. Since, for a function $f^* \in \mathbb{C}[0, \infty)$, and $T < \infty$, $\|f^*\|_T \leq \|f^*\|_{\mathbb{L}^\infty(0, \infty)}$, part b follows immediately from the bound (5) of [8, Theorem 8.8]. Part c is elementary; however, a proof is provided in Appendix B. \square

Using I to denote the embedding from $\mathbb{H}^1(0, \infty)$ to $\mathbb{C}[0, \infty)$ as defined above, we define the space

$$\mathbb{Y} \doteq \{(x, f) \in \mathbb{R} \times \mathbb{H}^1(0, \infty) : I[f](0) = x \wedge 0\},$$

which will serve as the state space of the diffusion model.

COROLLARY 2.4. *\mathbb{Y} is a closed subspace of $\mathbb{R} \times \mathbb{H}^1(0, \infty)$ and hence, is a Polish space.*

PROOF. Note that the mapping $f^* \mapsto f^*(0)$ is continuous from $\mathbb{C}[0, \infty)$ to \mathbb{R} , and hence by part b of Lemma 2.3, the mapping that takes $f \in \mathbb{H}^1(0, \infty)$ to $I[f](0)$ is continuous from $\mathbb{H}^1(0, \infty)$ to \mathbb{R} . Since $x \mapsto x \wedge 0$ is also continuous on \mathbb{R} , the set \mathbb{Y} is the pre-image of the closed set $\{0\}$ under the continuous map $I[f](0) - x \wedge 0$, and hence is closed. \square

REMARK 2.5. Lemma 2.3 asserts that for every function $f \in \mathbb{H}^1(0, \infty)$, there exists a unique continuous function f^* on $[0, \infty)$ whose restriction to $(0, \infty)$ belongs to the equivalence class of f . This continuous representative is used to define the evaluation of f on $[0, \infty)$; in particular, we use $f(0)$ to denote the evaluation of f^* at 0. For ease of notation (and as is customary in the literature, see, e.g., [8, Remark 5, Section 8]), we denote the continuous representative of f again by f .

Using the notation of Remark 2.5, we can rewrite the state space \mathbb{Y} as

$$(2.17) \quad \mathbb{Y} \doteq \{(x, f) \in \mathbb{R} \times \mathbb{H}^1(0, \infty) : f(0) = x \wedge 0\}.$$

Let \mathbb{Y} be equipped with its Borel σ -algebra $\mathcal{B}(\mathbb{Y})$. We now introduce the diffusion model SPDE, whose form is motivated by the discussion in Section 2.2.1. Specifically, equation (2.19) for Z follows from (2.13) and (2.12), equation (2.21) for X follows from (2.14), and the boundary condition (2.20) coincides with (2.15).

DEFINITION 2.6 (Diffusion Model SPDE). Let $Y_0 = (X_0, Z_0(\cdot))$ be a \mathbb{Y} -valued random element defined on $(\Omega, \mathcal{F}, \mathbb{P})$, independent of B and \mathcal{M} . A continuous \mathbb{Y} -valued stochastic process $Y = \{(X(t), Z(t, \cdot)); t \geq 0\}$ is said to be a solution of the *diffusion model SPDE* with initial condition Y_0 if

1. Y is $\{\mathcal{F}_t^{Y_0}\}$ -adapted, where $\mathcal{F}_t^{Y_0} \doteq \mathcal{F}_t \vee \sigma(Y_0)$;
2. $Y(0) = Y_0$, \mathbb{P} -almost surely;
3. \mathbb{P} -almost surely, $\partial_r Z(\cdot, \cdot) : (0, \infty) \times (0, \infty) \mapsto \mathbb{R}$ is locally integrable and for every $t > 0$, there exists (a unique) $F_t^* \in \mathbb{C}[0, \infty)$ such that the function

$$(2.18) \quad r \mapsto \int_0^t \partial_r Z(s, r) ds$$

is equal to F_t^* a.e. on $(0, \infty)$. Again, with a slight abuse of notation, for $r \geq 0$ (and in particular, $r = 0$) we denote by $\int_0^t \partial_r Z(s, r) ds$ the evaluation of the continuous representative F_t^* at r .

4. \mathbb{P} -almost surely, Z satisfies

$$(2.19) \quad \begin{aligned} Z(t, r) = & Z_0(r) + \int_0^t \partial_r Z(s, r) ds - \mathcal{M}_t(\vartheta^r) \\ & + \overline{G}(r) \left\{ Z(t, 0) - Z_0(0) - \int_0^t \partial_r Z(s, 0) ds + \mathcal{M}_t(\mathbf{1}) \right\}, \quad \forall t, r \geq 0, \end{aligned}$$

subject to the boundary condition

$$(2.20) \quad Z(t, 0) = X(t) \wedge 0, \quad \forall t \geq 0,$$

and X satisfies the stochastic equation

$$(2.21) \quad X(t) = X_0 + \sigma B(t) - \beta t - \mathcal{M}_t(\mathbf{1}) + \int_0^t \partial_r Z(s, 0) ds, \quad \forall t \geq 0.$$

Given B, \mathcal{M} as above, we say the diffusion model SPDE has a unique solution if for every initial condition Y_0 and every two solutions Y and \tilde{Y} with initial condition Y_0 , $\mathbb{P}\{Y(t) = \tilde{Y}(t); \forall t \geq 0\} = 1$.

2.3. Main Results. Our first result concerns uniqueness of the solution to the diffusion model SPDE. Recall that a Markov family $\{P^y; y \in \mathbb{Y}\}$ with the corresponding transition semigroup $\{\mathcal{P}_t; t \geq 0\}$ is called Feller if for every continuous and bounded function F , $\mathcal{P}_t[F]$ is a continuous function.

THEOREM 2.7. *Suppose Assumptions I and II hold. Then, for every \mathbb{Y} -valued random element Y_0 , there exists a unique solution Y to the diffusion model SPDE with initial condition Y_0 . Furthermore, if P^y is the law of the solution with initial condition $y \in \mathbb{Y}$, then $\{P^y; y \in \mathbb{Y}\}$ is a time-homogeneous Feller Markov family.*

The proof of Theorem 2.7 is given at the end of Section 3.5. Let $\{\mathcal{P}_t; t \geq 0\}$ be transition semigroup associated with the Markov family $\{P^y; y \in \mathbb{Y}\}$ of Theorem 2.7, and recall that a probability measure μ on $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$ is said to be an invariant distribution for $\{\mathcal{P}_t\}$ if and only if

$$(2.22) \quad \mu \mathcal{P}_t = \mu, \quad t \geq 0.$$

THEOREM 2.8. *Suppose Assumptions I and II are satisfied. Then the transition semigroup $\{\mathcal{P}_t; t \geq 0\}$ associated with the diffusion model SPDE has at most one invariant distribution.*

Theorem 2.8 is proved in Section 4.6.

REMARK 2.9. It is possible to establish existence of the invariant distribution of $\{\mathcal{P}_t\}$ under a finite $(2 + \epsilon)$ moment assumption, by an application of the Krylov-Bogoliubov theorem together with certain uniform bounds on the fluctuations of the number of jobs $\widehat{X}^{(N)}$ in the N server queue, obtained in [3, Corollary 5.5]. However, we omit the proof from this paper since we establish existence in a companion paper [3, Corollary 7.6] under only mild additional assumptions that, for example, require the service distribution to have a finite $(3 + \epsilon)$ moment. The proof of existence in [3] proceeds by showing that the sequence of stationary distributions $\{\widehat{\pi}^{(N)}\}$ of the state representation $\widehat{Y}^{(N)}$ of the GI/GI/N queue (described in Section 2.2.1) is tight, and that every subsequential limit of $\{\pi^{(N)}\}$ is an invariant distribution of $\{\mathcal{P}_t; t \geq 0\}$. Moreover, although the condition $\beta > 0$ is not necessary for the results of this paper, it is required to prove the existence of stationary distribution (see the proof of Proposition 3.1 in [3]). Thus we make the assumption $\beta > 0$ throughout this paper.

REMARK 2.10. Our key observation is that a choice of state in which one appends to \mathbb{R} , the state space of X , a function-valued process such as Z allows for the analysis of the process and its invariant distribution. However, it is worth emphasizing that the most convenient choice of function space for Z is not completely obvious. For example, Z could also be viewed as a continuous process taking values in the spaces $\mathbb{L}^2(0, \infty)$, $\mathbb{C}[0, \infty)$, $\mathbb{C}^1[0, \infty)$, or $\mathbb{W}^{1,1}(0, \infty)$, the Sobolev space of integrable functions with integrable weak derivatives on $(0, \infty)$. However, in the case of $\mathbb{L}^2(0, \infty)$ or $\mathbb{C}[0, \infty)$, X does not seem to admit a representation as a nice Itô process, and it is not clear if (X, Z) is a Feller process. On the other hand, although the choice of $\mathbb{C}^1[0, \infty)$ leads to a Feller process, it seems difficult to show uniqueness of the invariant distribution in this space. Lastly, when the state space of Z is chosen to be $\mathbb{W}^{1,1}(0, \infty)$, it is possible to show that (X, Z) is both a continuous homogeneous Feller process and has a unique invariant distribution (albeit the latter only under more restrictive assumptions on G). However, in this case, it does not seem easy to establish convergence of the centered scaled marginals $\widehat{Z}^N(t)$ to $Z(t)$, which is a key step in [3] that is used to identify the limit of the sequence of scaled N -server stationary distributions. The choice of the space $\mathbb{H}^1(0, \infty)$ for Z allows us to establish both the Feller property and uniqueness of the invariant distribution for a large class of service distributions (including those listed below Remark 2.2), and the Hilbert structure of $\mathbb{H}^1(0, \infty)$ is exploited in [3, Lemma 7.5 and Proposition 7.2] to establish finite-dimensional convergence of $\widehat{Z}^N(t)$ to $Z(t)$ when the initial conditions converge in a suitable sense. Another potential state space for Z is $\{f \in \mathbb{C}[0, \infty) : \exists f' \in \mathbb{L}^2[0, \infty) \text{ s.t. } f(t) = \int_0^t f'(s) ds \text{ for all } t \in [0, \infty)\}$. However, we found the convergence calculations less tedious with the choice of $\mathbb{H}^1(0, \infty)$.

3. An Explicit Solution to the SPDE. The goal of this section is to prove Theorem 2.7. We start by establishing existence and uniqueness of a solution to the diffusion model SPDE. First, in Section 3.1, we state a number of basic results that are required to define a candidate solution,

which we call the diffusion model. In Section 3.2, we provide an explicit construction of this \mathbb{Y} -valued stochastic process. In Section 3.3, we verify that Y is indeed a solution to the diffusion model SPDE and in Section 3.4, prove that it is the unique solution. Finally, in Section 3.5 we show that the diffusion model is a time-homogeneous Feller Markov process.

3.1. Preliminaries. In this section we establish regularity properties of various objects that arise in the analysis of the SPDE. As these results are not the central contribution of the paper, most of their proofs are relegated to Section 5.

3.1.1. Properties of the Martingale Measure. Recall the definition given in Section 2.2 of the continuous martingale measure \mathcal{M} and stochastic integral $\mathcal{M}_t(\varphi)$. We now define two families of operators that allow us to represent some relevant quantities in a more succinct manner. Consider the family $\{\Psi_t, t \geq 0\}$ of operators that map functions on $[0, \infty)$ to functions on $[0, \infty) \times [0, \infty)$, and are defined as follows: for every $t \geq 0$,

$$(3.1) \quad (\Psi_t f)(x, s) \doteq f(x + (t - s)^+) \frac{\overline{G}(x + (t - s)^+)}{\overline{G}(x)}, \quad (x, s) \in [0, \infty) \times [0, \infty).$$

In particular, for any bounded function f on $[0, \infty)$, Ψ_t satisfies

$$\sup_{t \geq 0} \|\Psi_t f\|_\infty \leq \|f\|_\infty,$$

and hence, Ψ_t maps the space $\mathbb{C}_b[0, \infty)$ of continuous bounded functions on $[0, \infty)$ to the space $\mathbb{C}_b([0, \infty) \times [0, \infty))$ of continuous bounded functions on $[0, \infty) \times [0, \infty)$. Next, consider the family of operators $\{\Phi_t, t \geq 0\}$ which are defined as follows: for every $t \geq 0$, let

$$(3.2) \quad (\Phi_t f)(x) \doteq f(x + t) \frac{\overline{G}(x + t)}{\overline{G}(x)}, \quad x \in [0, \infty),$$

for any function f on $[0, \infty)$. The operator Φ_t clearly maps the space $\mathbb{C}_b[0, \infty)$ into itself.

Note that the family of functions $\{\vartheta^r; r \geq 0\}$ defined in (2.10) has the representation

$$(3.3) \quad \vartheta^r = \Phi_r \mathbf{1}, \quad r \geq 0.$$

Also, $\Phi_0 f = f$ and the family $\{\Phi_t, t \geq 0\}$ satisfies the following semigroup property:

$$\Phi_t \Phi_s = \Phi_{t+s}, \quad t, s \geq 0.$$

Furthermore, for every function f defined on $[0, \infty)$ and $s, t > 0$,

$$(3.4) \quad (\Psi_s \Phi_t f)(x, v) = (\Psi_{s+t} f)(x, v), \quad (x, v) \in [0, \infty) \times [0, s].$$

Next, we define a family of stochastic convolution integrals: for $t \geq 0$ and $f \in \mathbb{C}_b[0, \infty)$,

$$(3.5) \quad \mathcal{H}_t(f) \doteq \mathcal{M}_t(\Psi_t f) = \int \int_{[0, \infty) \times [0, t]} f(x + t - s) \frac{\overline{G}(x + t - s)}{\overline{G}(x)} \mathcal{M}(dx, ds).$$

Recall that a stochastic process or random field $\{\xi_1(t); t \in \mathcal{T}\}$ with index set \mathcal{T} is called a modification of another stochastic process or random field $\{\xi_2(t); t \in \mathcal{T}\}$ defined on a common probability space, if for every $t \in \mathcal{T}$, $\xi_1(t) = \xi_2(t)$, almost surely. In contrast, ξ_1 and ξ_2 are said to be indistinguishable if $\mathbb{P}\{\xi_1(t) = \xi_2(t) \text{ for all } t \in \mathcal{T}\} = 1$. Lemma 3.1 and Proposition 3.3 below show that we can choose suitably regular modifications of certain stochastic integrals.

LEMMA 3.1. *Suppose Assumption I holds. Then, $\{\mathcal{M}_t(\Phi_r \mathbf{1}); t, r \geq 0\}$, $\{\mathcal{M}_t(\Psi_{t+r} \mathbf{1}); t, r \geq 0\}$ and $\{\mathcal{M}_t(\Psi_{t+r} h); t, r \geq 0\}$ have modifications that are jointly continuous on $[0, \infty) \times [0, \infty)$.*

PROOF. The proof is deferred to Section 5.1.1. \square

REMARK 3.2. By $\mathcal{M}_t(\Phi_r \mathbf{1})$, $\mathcal{M}_t(\Psi_{t+r} \mathbf{1})$ and $\mathcal{M}_t(\Psi_{t+r} h)$, we always denote the jointly continuous modification. Note that, by substituting $r = 0$ in Lemma 3.1, this also implies the continuity of the stochastic processes $t \mapsto \mathcal{M}_t(\mathbf{1})$, $t \mapsto \mathcal{H}_t(\mathbf{1})$ and $t \mapsto \mathcal{H}_t(h)$.

PROPOSITION 3.3. *Let Assumption I and Assumption II hold. Then $\{\mathcal{M}_t(\Psi_{t+} \mathbf{1}); t \geq 0\}$ has a continuous $\mathbb{H}^1(0, \infty)$ -valued modification. Also, almost surely, for every $t \geq 0$, the function $r \mapsto \mathcal{M}_t(\Psi_{t+r} \mathbf{1})$ has weak derivative $-\mathcal{M}_t(\Psi_{t+r} h)$ on $(0, \infty)$.*

PROOF. The proof is deferred to the end of Section 5.1.2. \square

REMARK 3.4. By the continuous embedding result of Lemma 2.3.b, given a real-valued function f on $[0, \infty) \times (0, \infty)$ such that $t \mapsto f(t, \cdot)$ is continuous from $[0, \infty)$ to $\mathbb{H}^1(0, \infty)$, the mapping $(t, r) \mapsto f(t, r)$ has a representative that is a jointly continuous function on $[0, \infty) \times [0, \infty)$. Therefore, the continuous modification of $\{\mathcal{M}_t(\Psi_{t+}(\mathbf{1}); t \geq 0\}$ in Proposition 3.3 is indistinguishable from the jointly continuous modification of $\{\mathcal{M}_t(\Psi_{t+r}(\mathbf{1}); t, r \geq 0\}$ in Remark 3.2.

Next, we establish a simple relation that is used in the proof of Theorem 2.7.

LEMMA 3.5. *Suppose Assumption I holds. Then, almost surely*

$$(3.6) \quad \mathcal{M}_t(\Psi_{t+r} \mathbf{1}) = \mathcal{M}_t(\Phi_r \mathbf{1}) - \int_0^t \mathcal{M}_s(\Psi_{s+r} h) ds, \quad \forall t, r \geq 0.$$

PROOF. The proof is deferred to Section 5.1.3. \square

Setting $r = 0$ in (3.6), we see that, almost surely,

$$(3.7) \quad \mathcal{H}_t(\mathbf{1}) = \mathcal{M}_t(\mathbf{1}) - \int_0^t \mathcal{H}_s(h) ds, \quad t \geq 0.$$

3.1.2. *An Auxiliary Mapping.* Next, we introduce an auxiliary mapping that appears in the explicit construction of the diffusion model (see Definition 3.9). For every $t \geq 0$, define

$$(3.8) \quad (\Gamma_t \kappa)(r) \doteq \overline{G}(r) \kappa(t) - \int_0^t \kappa(s) g(t+r-s) ds, \quad r \in [0, \infty),$$

for $\kappa \in \mathbb{C}[0, \infty)$. The following lemma establishes useful properties of the mapping Γ_t .

LEMMA 3.6. *Under Assumptions I and II, the following assertions hold:*

a. *For every $t \geq 0$ and $\kappa \in \mathbb{C}[0, \infty)$, the function $\Gamma_t \kappa$ lies in $\mathbb{H}^1(0, \infty) \cap \mathbb{C}^1[0, \infty)$ and has derivative*

$$(3.9) \quad (\Gamma_t \kappa)'(r) = -g(r) \kappa(t) - \int_0^t \kappa(s) g'(t+r-s) ds, \quad r \in [0, \infty).$$

- b. For every $t \geq 0$, the mapping $\Gamma_t : \mathbb{C}[0, \infty) \rightarrow \mathbb{H}^1(0, \infty)$ is continuous.
- c. For every $\kappa \in \mathbb{C}[0, \infty)$, the mapping $t \mapsto \Gamma_t \kappa$ from $[0, \infty)$ to $\mathbb{H}^1(0, \infty)$ is continuous.

PROOF. The proof of Lemma 3.6 can be found in Section 5.2.1. \square

3.2. The Diffusion Model. Here, we explicitly construct our proposed diffusion model $Y = (X, Z)$. In the next two subsections we show that Y is indeed the unique solution to the diffusion model SPDE. The X -component of this process is defined in terms of the (deterministic) centered many-server (CMS) mapping introduced in [29], and recalled below as Definition 3.8.

The first assertion of Lemma 3.7 below, which characterizes the CMS mapping, and shows it is continuous, can be deduced from a more general result established in [29] (see Proposition 7.3 and the proof of Lemma 7.4 therein). Since the proof of this characterization is simpler in our context (because we only consider the so-called critical regime) for completeness, we include a direct proof below. Recall that $\mathbb{C}^0[0, \infty)$ denotes the set of functions $f \in \mathbb{C}[0, \infty)$ with $f(0) = 0$.

LEMMA 3.7. *Given $(\eta, x_0, \zeta) \in \mathbb{C}^0[0, \infty) \times \mathbb{R} \times \mathbb{C}[0, \infty)$ with $\zeta(0) = x_0 \wedge 0$, there exists a unique pair $(\kappa, x) \in \mathbb{C}^0[0, \infty) \times \mathbb{C}[0, \infty)$ that satisfies the following equations: for every $t \in [0, \infty)$,*

$$(3.10) \quad x(t) \wedge 0 = \zeta(t) + \kappa(t) - \int_0^t g(t-s)\kappa(s)ds,$$

$$(3.11) \quad \kappa(t) = \eta(t) - x^+(t) + x_0^+.$$

Furthermore, the mapping $\Lambda : \mathbb{C}^0[0, \infty) \times \mathbb{R} \times \mathbb{C}[0, \infty) \mapsto \mathbb{C}^0[0, \infty) \times \mathbb{C}[0, \infty)$ that takes (η, x_0, ζ) to (κ, x) is continuous and non-anticipative, that is, for every $T \in (0, \infty)$ and $(\kappa_i, x_i) = \Lambda(\eta_i, x_0, \zeta_i)$, $i = 1, 2$, if (η_1, ζ_1) and (η_2, ζ_2) are equal on $[0, T]$, then (κ_1, x_1) coincides with (κ_2, x_2) on $[0, T]$.

PROOF. Fix $(\eta, x_0, \zeta) \in \mathbb{C}^0[0, \infty) \times \mathbb{R} \times \mathbb{C}[0, \infty)$ with $\zeta(0) = x_0 \wedge 0$, and set

$$(3.12) \quad r(t) \doteq \zeta(t) + \eta(t) - \int_0^t g(t-s)\eta(s)ds + \overline{G}(t)x_0^+, \quad t \geq 0.$$

Then, substituting κ from (3.11) into (3.10), it is straightforward to see that (κ, x) satisfy (3.10) and (3.11) if and only if x satisfies the Volterra equation

$$(3.13) \quad x(t) = r(t) + \int_0^t g(t-s)x^+(s)ds, \quad t \geq 0,$$

and κ satisfies (3.11). However, since $F(x) = x^+$ is Lipschitz and the assumptions on (η, x_0, ζ) imply that r is continuous and $r(0) = x_0$, there exists a unique solution \bar{x} to (3.13) (Theorem 3.2.1 of [10] shows that a unique solution exists on a finite interval, while Theorem 3.3.6 of [10] ensures that the solution can be extended to the whole interval $[0, \infty)$). Defining $\bar{\kappa}$ as in (3.11), with x replaced by \bar{x} , it follows that $(\bar{\kappa}, \bar{x})$ is the unique solution to the equations (3.10)-(3.11) associated with (η, x_0, ζ) . Let Λ denote the map that takes (η, x_0, ζ) to this unique solution.

The continuity of Λ follows from Proposition 7.3 in [29]. To prove the non-anticipative property, for $i = 1, 2$, define r_i by (3.12) with η and ζ replaced by η_i and ζ_i . We need to show that for every $t \in [0, T]$, if (η_1, ζ_1) and (η_2, ζ_2) agree on $[0, t]$, we have $r_1(t) = r_2(t)$. Subtracting equation (3.13) with x and r replaced by x_i and r_i , $i = 1, 2$, respectively, and defining $\Delta x = x_1 - x_2$, we have

$$\Delta x(t) = \int_0^t g(t-s) (x_1^+(s) - x_2^+(s)) ds, \quad t \in [0, T].$$

Using the fact that the map $F(x) = x^+$ is Lipschitz with constant 1, we have

$$|\Delta x(t)| \leq \int_0^t g(t-s)|x_1^+(s) - x_2^+(s)|ds \leq \int_0^t g(t-s)|\Delta x(s)|ds \quad t \in [0, T].$$

Since $\Delta x(0) = 0$, Gronwall's inequality implies $\Delta x(t) = 0$ for $t \in [0, T]$. Then, because κ_i satisfies (3.11) with η and x replaced by η_i and x_i , respectively, we also have $\kappa_1(t) = \kappa_2(t)$ for $t \in [0, T]$. \square

DEFINITION 3.8 (Centered Many-Server Mapping). Equations (3.10) and (3.11) are called the centered many-server (CMS) equations associated with (η, x_0, ζ) , and the mapping Λ that takes (η, x_0, ζ) to the unique solution (κ, x) of the CMS equations associated with (η, x_0, ζ) is called the centered many-server (CMS) Mapping.

Recall that B is a Brownian motion and for fixed $\sigma, \beta > 0$, define

$$(3.14) \quad E(t) \doteq \sigma B(t) - \beta t, \quad t \geq 0.$$

DEFINITION 3.9 (Diffusion Model). For every \mathbb{Y} -valued random element $Y_0 = (X_0, Z_0)$, the diffusion model $Y^{Y_0} = \{Y^{Y_0}(t); t \geq 0\}$ with initial condition Y_0 is defined by $Y^{Y_0}(t) = (X(t), Z(t, \cdot))$, where

$$(3.15) \quad (K, X) \doteq \Lambda(E, X_0, Z_0 - \mathcal{H}(\mathbf{1})),$$

and for every $t, r \geq 0$,

$$(3.16) \quad Z(t, r) \doteq Z_0(t+r) - \mathcal{M}_t(\Psi_{t+r}\mathbf{1}) + \Gamma_t K(r),$$

where $\{\Psi_t; t \geq 0\}$ and $\{\Gamma_t; t \geq 0\}$ are given by (3.1) and (3.8), respectively.

Deterministic initial conditions will be denoted by lower case: $y = (x_0, z_0) \in \mathbb{Y}$. Also, when the initial condition Y_0 is clear from the context, we will often not mention it explicitly, and also omit the superscript Y_0 and just use Y to denote the diffusion model.

REMARK 3.10. Note that E and $\mathcal{H}(\mathbf{1})$ have continuous sample paths and $E(0) = \mathcal{H}_0(\mathbf{1}) = 0$. Also, for every \mathbb{Y} -valued random element (X_0, Z_0) , by Lemma 2.3.a and Remark 2.5, Z_0 has a representative (also denoted by Z_0) which is continuous on $[0, \infty)$ and satisfies $Z_0(0) = X_0 \wedge 0$ by the definition of \mathbb{Y} . Therefore, almost surely, $(E, x_0, Z_0 - \mathcal{H}(\mathbf{1}))$ lies in the domain of the CMS mapping Λ . Hence, (K, X) in (3.15) is well defined and (by Lemma 3.7) satisfies a.s.,

$$(3.17) \quad X(t) \wedge 0 = Z_0(t) - \mathcal{H}_t(\mathbf{1}) + K(t) - \int_0^t g(t-s)K(s)ds,$$

$$(3.18) \quad K(t) = \sigma B(t) - \beta t - X^+(t) + X_0^+,$$

for $t \geq 0$. Also, almost surely for every $t \geq 0$, $Z_0 \in \mathbb{H}^1(0, \infty)$ implies $Z_0(t + \cdot)$ lies in $\mathbb{H}^1(0, \infty)$ (see Lemma 2.3.c), the continuity of K and Lemma 3.6.a imply $\Gamma_t K(\cdot) \in \mathbb{H}^1(0, \infty)$, and Proposition 3.3 implies $r \mapsto \mathcal{M}_t(\Psi_{t+r}\mathbf{1})$ lies in $\mathbb{H}^1(0, \infty)$. Thus, $Z(t, \cdot)$ in (3.16) also lies in $\mathbb{H}^1(0, \infty)$ and furthermore, by Lemma 2.3.a, has a continuous modification, so $Z(t, r)$ in (3.16) is well defined for $r \in [0, \infty)$.

3.3. Existence of a Solution. We now show that the diffusion model defined in Section 3.2 is indeed a solution of the diffusion model SPDE. Throughout this section, we fix $Y_0 = (X_0, Z_0) \in \mathbb{Y}$ and let $Y = Y^{Y_0}$ be the diffusion model with initial condition Y_0 , as specified in Definition 3.9.

PROPOSITION 3.11. *Suppose Assumptions I and II hold. Then the diffusion model $Y = (X, Z)$ with initial condition Y_0 satisfies the following properties:*

- a. *Almost surely, the sample paths of $\{Z(t, \cdot); t \geq 0\}$ are $\mathbb{H}^1(0, \infty)$ -valued and continuous, and for every $t \geq 0$, the weak derivative $\partial_r Z(t, \cdot)$ of $Z(t, \cdot)$ satisfies for a.e. $r \in (0, \infty)$,*

$$(3.19) \quad \partial_r Z(t, r) = Z'_0(t + r) + \mathcal{M}_t(\Psi_{t+r}h) - g(r)K(t) - \int_0^t K(s)g'(t + r - s)ds.$$

- b. *Almost surely, $Z(t, 0) = X(t) \wedge 0$, for all $t \geq 0$.*
c. *$\{Y(t); t \geq 0\}$ is an almost surely continuous \mathbb{Y} -valued process.*

PROOF. For part a, we look at each term in the definition of Z in (3.16) separately. Since $Z_0 \in \mathbb{H}^1(0, \infty)$, the translation mapping $t \mapsto Z_0(t + \cdot)$ is continuous in $\mathbb{H}^1(0, \infty)$ by Lemma 2.3.c and for each $t > 0$, $Z_0(t + \cdot)$ has weak derivative $Z'_0(t + \cdot)$. For the second term, by Proposition 3.3, $\{\mathcal{M}_t(\Psi_{t+} \mathbf{1}); t \geq 0\}$ is a continuous $\mathbb{H}^1(0, \infty)$ -valued process and almost surely, for every $t \geq 0$, the function $r \mapsto \mathcal{M}_t(\Psi_{t+r} \mathbf{1})$ has weak derivative $r \mapsto -\mathcal{M}_t(\Psi_{t+r}h)$. Finally for the third term, since the range of the CMS map Λ lies in $\mathbb{C}^0[0, \infty) \times \mathbb{C}[0, \infty)$ (see Definition 3.8 and Lemma 3.7), almost surely, the process K defined in (3.15) is continuous. Therefore, by Lemma 3.6.c, $\{(\Gamma_t K); t \geq 0\}$ is a continuous $\mathbb{H}^1(0, \infty)$ -valued process. Also, by (3.9), for every $t \geq 0$, $r \mapsto \Gamma_t K(r)$ has weak derivative $-g(r)K(t) - \int_0^t K(s)g'(t + r - s)ds$. This completes the proof of part a.

Next, for part b, substituting $r = 0$ in (3.16) and the definition (3.8) of Γ_t , and using the identity $\mathcal{H}_t(\mathbf{1}) = \mathcal{M}_t(\Psi_t \mathbf{1})$ from (3.5), we obtain

$$Z(t, 0) = Z_0(t) - \mathcal{H}_t(\mathbf{1}) + K(t) - \int_0^t K(s)g(t - s)ds.$$

The assertion in b then follows from equation (3.17).

Finally, since the range of Λ lies in $\mathbb{C}^0[0, \infty) \times \mathbb{C}[0, \infty)$ (see Definition 3.7 and Lemma 3.7), by (3.15) $\{X(t); t \geq 0\}$ is a.s. continuous. Along with parts a and b above, this proves part c. \square

Next, we show that Z satisfies the regularity condition required in part 3 of Definition 2.6.

LEMMA 3.12. *Suppose Assumptions I and II hold, and let $Y = (X, Z)$ be the diffusion model with initial condition Y_0 . Then, almost surely, $(s, r) \mapsto \partial_r Z(s, r)$ is locally integrable on $(0, \infty) \times (0, \infty)$ and for every $t \geq 0$, there exists a continuous function F_t^* on $[0, \infty)$ such that the function $r \mapsto \int_0^t \partial_r Z(s, r)ds$ is equal to F_t^* almost everywhere on $(0, \infty)$. Moreover, for every $t, r \geq 0$*

$$(3.20) \quad \int_0^t \partial_r Z(s, r)ds = Z_0(t + r) - Z_0(r) + \mathcal{M}_t(\Phi_r \mathbf{1}) - \mathcal{M}_t(\Psi_{t+r} \mathbf{1}) - \int_0^t K(s)g(t + r - s)ds.$$

REMARK 3.13. Recall again that for $r \geq 0$ (and, in particular, $r = 0$), $\int_0^t \partial_r Z(s, r)ds$, $Z_0(t + r)$ and $Z_0(r)$ in the above identity denote the evaluation of their corresponding continuous representative at r .

PROOF OF LEMMA 3.12. By Proposition 3.11.a, almost surely for every $s \geq 0$, the weak derivative $\partial_r Z(s, \cdot)$ of $r \mapsto Z(s, r)$ exists and is given by (3.19). We prove the claims separately for each term on the right-hand side of (3.19). First, since $Z_0(\cdot) \in \mathbb{H}^1(0, \infty)$, the mapping $(s, r) \mapsto Z'_0(r + s)$ is clearly locally integrable on $(0, \infty) \times (0, \infty)$ and for every $t \geq 0$, and almost every $r \in (0, \infty)$,

$$(3.21) \quad \int_0^t Z'_0(s + r) ds = Z_0(t + r) - Z_0(r).$$

Moreover, for every $t \geq 0$, since $Z_0(\cdot) \in \mathbb{H}^1(0, \infty)$, $Z_0(t + \cdot)$ also lies in $\mathbb{H}^1(0, \infty)$, and therefore by Lemma 2.3.a, there exists a continuous function on $[0, \infty)$, that is equal to $Z_0(t + r) - Z_0(r)$ almost everywhere on $(0, \infty)$.

Next, Lemma 3.1 implies that almost surely, $(s, r) \mapsto \mathcal{M}_s(\Psi_{s+r}h)$ is jointly continuous and hence, locally integrable on $(0, \infty) \times (0, \infty)$. Also, for every $t \geq 0$, by (3.6) of Lemma 3.5,

$$(3.22) \quad \int_0^t \mathcal{M}_s(\Psi_{s+r}h) ds = \mathcal{M}_t(\Phi_r \mathbf{1}) - \mathcal{M}_t(\Psi_{t+r} \mathbf{1}), \quad r \geq 0.$$

By Lemma 3.1, $r \mapsto \mathcal{M}_t(\Phi_r \mathbf{1}) - \mathcal{M}_t(\Psi_{t+r} \mathbf{1})$ is continuous on $[0, \infty)$.

Finally, it follows from the continuity of K , g and g' (see Assumption I) that the mapping $(s, r) \mapsto g(r)K(s) + \int_0^s K(v)g'(s + r - v)dv$ is jointly continuous and hence, locally integrable on $(0, \infty) \times (0, \infty)$. Also, for every $t \geq 0$, using Fubini's theorem,

$$(3.23) \quad \begin{aligned} \int_0^t \left(g(r)K(s) + \int_0^s K(v)g'(s + r - v)dv \right) ds &= g(r) \int_0^t K(s)ds + \int_0^t \int_v^t K(v)g'(s + r - v)ds dv \\ &= \int_0^t K(s)g(t + r - s)ds. \end{aligned}$$

Again, by the continuity of K and g , the mapping $r \mapsto \int_0^t K(s)g(t + r - s)ds$ is continuous on $[0, \infty)$. Equation (3.20) then follows from equations (3.19)-(3.23). \square

We now obtain an alternative characterization of the diffusion model SPDE (2.19)-(2.21).

LEMMA 3.14. *Given a \mathbb{Y} -valued random element $Y_0 = (X_0, Z_0(\cdot))$, let $\{Y(t) = (X(t), Z(t, \cdot)); t \geq 0\}$ be a continuous \mathbb{Y} -valued process that satisfies conditions 1-3 of Definition 2.6, and suppose (X, Z) satisfies equations (2.20) and (2.21). Then, (X, Z) satisfies equation (2.19) if and only if for all $t, r \geq 0$,*

$$(3.24) \quad Z(t, r) = Z_0(r) + \int_0^t \partial_r Z(s, r) ds - \mathcal{M}_t(\Phi_r \mathbf{1}) + \overline{G}(r)K(t),$$

with $K(t) = \sigma B(t) - \beta t - X^+(t) + X_0^+$.

PROOF. Comparing (3.24) and (2.19), and recalling that $\vartheta^r = \Phi_r \mathbf{1}$, it is clear that these two equations are equivalent if and only if for every $t \geq 0$, the following identity holds:

$$(3.25) \quad K(t) = Z(t, 0) - Z_0(0) - \int_0^t \partial_r Z(s, 0) ds + \mathcal{M}_t(\mathbf{1}).$$

On the other hand, by (2.20) and (2.21) and the definition of K given above, for $t \geq 0$, we have

$$\begin{aligned} Z(t, 0) &= X(t) - X^+(t) \\ &= X_0 + \sigma B(t) - \beta t - \mathcal{M}_t(\mathbf{1}) + \int_0^t \partial_r Z(s, 0) ds - X^+(t) \\ &= K(t) - \mathcal{M}_t(\mathbf{1}) + \int_0^t \partial_r Z(s, 0) ds + X_0 - X_0^+. \end{aligned}$$

Equation (2.20) (for $t = 0$) also implies that $Z_0(0) = Z(0, 0) = X(0) \wedge 0 = X_0 - X_0^+$. When substituted into the last display, (3.25) follows. \square

We now show that the process specified in Definition 3.9 satisfies the diffusion model SPDE.

PROPOSITION 3.15. *Suppose Assumptions I and II are satisfied, and given a \mathbb{Y} -valued random element Y_0 , let $Y = \{Y(t); t \geq 0\}$ be the diffusion model with initial condition Y_0 specified in Definition 3.9. Then Y is a solution of the diffusion model SPDE with initial condition Y_0 .*

PROOF. By Proposition 3.11.c, Y is a continuous \mathbb{Y} -valued process. We show that it satisfies conditions 1-4 of Definition 2.6. Condition 2 holds because $Y(0) = Y_0$ by definition, and condition 3 follows from Lemma 3.12. Next, we verify condition 1 of Definition 2.6 by showing that Y is $\{\mathcal{F}_t^{Y_0}\}$ -adapted. Fix $t \geq 0$, and note that the stopped processes $\{E(s \wedge t); s \geq 0\}$ and $\{\mathcal{H}_{s \wedge t}(\mathbf{1}); s \geq 0\}$ agree with E and $\mathcal{H}(\mathbf{1})$, respectively, on $[0, t]$. Hence, by definition (3.15) of (K, X) and the non-anticipative property of Λ proved in Lemma 3.7, we have

$$(K(\cdot \wedge t), X(\cdot \wedge t)) = \Lambda(E(\cdot \wedge t), X_0, Z_0 - \mathcal{H}_{\cdot \wedge t}(\mathbf{1})).$$

Clearly, $\{E(s \wedge t); s \in [0, \infty)\}$ and $\{\mathcal{H}_{s \wedge t}(\mathbf{1}), s \in [0, \infty)\}$ are \mathcal{F}_t -measurable, while X_0 and Z_0 are $\sigma(Y_0)$ -measurable. Therefore, by the continuity, and hence, measurability of the mapping Λ proved in Lemma 3.7, $X(t)$ and $K(\cdot \wedge t)$ are $\mathcal{F}_t^{Y_0}$ -measurable. Moreover, from the definition of the mapping Γ_t , for every κ and $r \geq 0$, the value of $(\Gamma_t \kappa)(r)$ does not depend on the values of κ outside the interval $[0, t]$, and hence, $\Gamma_t K = \Gamma_t K(\cdot \wedge t)$. On the other hand, Γ_t is continuous by Lemma 3.6.c, and hence $\Gamma_t K(\cdot \wedge t)$ is also $\mathcal{F}_t^{Y_0}$ -measurable. Also, $\mathcal{M}_t(\Psi_{t+} \mathbf{1})$ is \mathcal{F}_t -measurable and $Z_0(t + \cdot)$ is $\sigma(Y_0)$ -measurable by definition. Therefore, $Z(t, \cdot)$ defined in (3.16) is $\mathcal{F}_t^{Y_0}$ -measurable. Consequently, Y is $\{\mathcal{F}_t^{Y_0}\}$ -adapted, and condition 1 follows.

We now turn to the proof of condition 4. By Lemma 3.14 it suffices to show that X and Z satisfy equations (3.24), (2.20) and (2.21). First, substituting $\int_0^t \partial_r Z(s, r) ds$ from (3.20), and using the definition (3.8) of Γ_t , the right-hand side of (3.24) is equal to

$$\begin{aligned} &Z_0(r) + Z_0(t + r) - Z_0(r) + \mathcal{M}_t(\Phi_r \mathbf{1}) - \mathcal{M}_t(\Psi_{t+r} \mathbf{1}) \\ &\quad - \int_0^t K(s) g(t + r - s) ds - \mathcal{M}_t(\Phi_r \mathbf{1}) + \overline{G}(r) K(t) \\ &= Z_0(t + r) - \mathcal{M}_t(\Psi_{t+r} \mathbf{1}) + \Gamma_t K(r). \end{aligned}$$

By (3.16), this is equal to $Z(t, r)$, which proves (3.24). Moreover, Proposition 3.11.b shows that the relation $X(t) \wedge 0 = Z(t, 0)$ in (2.20) holds. Combining this relation with the expression for $X^+(t)$ from (3.18), we have almost surely, for every $t \geq 0$,

$$X(t) = X^+(t) + X(t) \wedge 0 = X_0^+ + \sigma B(t) - \beta t - K(t) + Z(t, 0).$$

Together with the expression for $Z(t, 0)$ in (3.16) and for Γ_t in (3.8), both with $r = 0$, this implies

$$(3.26) \quad X(t) = X_0^+ + \sigma B(t) - \beta t + Z_0(t) - \mathcal{H}_t(\mathbf{1}) - \int_0^t K(s)g(t-s)ds,$$

whereas substituting $r = 0$ in (3.20), we have almost surely, for every $t \geq 0$,

$$(3.27) \quad \int_0^t \partial_r Z(s, 0)ds = Z_0(t) - Z_0(0) + \mathcal{M}_t(\mathbf{1}) - \mathcal{H}_t(\mathbf{1}) - \int_0^t K(s)g(t-s)ds,$$

where we have used the identities $\mathcal{H}_t(\mathbf{1}) = \mathcal{M}_t(\Psi_t \mathbf{1})$ and $\Phi_0 \mathbf{1} = \mathbf{1}$. Equation (2.21) then follows from (3.27), (3.26) and the relation $Z_0(0) = X(0) \wedge 0$. This completes the proof. \square

3.4. Uniqueness. Here, we show that the diffusion model SPDE has a unique solution. We first establish uniqueness of the weak solution to a transport equation within a certain class of functions.

LEMMA 3.16. *Suppose Assumption I holds, and a function $F \in \mathbb{C}[0, \infty)$ with $F(0) = 0$ is given. Let $\xi : [0, \infty) \times [0, \infty) \mapsto \mathbb{R}$ be a function that satisfies the following two properties:*

1. *The mapping $t \mapsto \xi(t, \cdot)$ lies in $\mathbb{C}([0, \infty); \mathbb{H}^1(0, \infty))$. For each $t > 0$, let $\partial_r \xi(t, \cdot)$ denote the weak derivative of $\xi(t, \cdot)$.*
2. *The function $\partial_r \xi : (0, \infty) \times (0, \infty) \mapsto \mathbb{R}$ is locally integrable.*

Then ξ satisfies the equation

$$(3.28) \quad \xi(t, r) = \int_0^t \partial_r \xi(s, r)ds + \overline{G}(r)F(t), \quad \text{a.e. } r \in (0, \infty),$$

for every $t \geq 0$ if and only if

$$(3.29) \quad \xi(t, r) = \Gamma_t F(r), \quad t, r \geq 0,$$

with Γ_t as defined in (3.8).

PROOF. This result would be standard if ξ and F were continuously differentiable functions. In that case, (3.28) would reduce to the classical inhomogeneous transport equation $\partial_t \xi(t, r) = \partial_r \xi(t, r) + \overline{G}(r)F'(t)$ with initial condition $\xi(0, \cdot) \equiv 0$, whose unique solution is [15, Section 2.1.2]:

$$\xi(t, r) = \int_0^t G(t-s+r)F'(s)ds = \overline{G}(r)F(t) - \int_0^t F(s)g(t-s+r)ds = \Gamma_t F(r).$$

While there are several related results, there appears to be no readily quotable result for the class of ξ and F mentioned above. Thus, for completeness, we include a proof in Section 5.2.2. \square

PROPOSITION 3.17. *Suppose Assumptions I and II hold. Then, for every \mathbb{Y} -valued random element $Y_0 = (X_0, Z_0)$, there is at most one solution of the diffusion model SPDE of Definition 2.6 with initial condition Y_0 .*

PROOF. For $i = 1, 2$, let $Y_i = (X_i, Z_i)$ be a solution to the diffusion model SPDE with initial condition Y_0 , and define $K_i(t) \doteq \sigma B(t) - \beta t - X_i^+(t) + X_0^+$. We need to show that $\mathbb{P}\{Y_1(t) = Y_2(t); \forall t \geq 0\} = 1$. Denote $\Delta H = H_1 - H_2$ for $H = Y, X, Z, K$, and note that $\Delta K(t) = X_2^+(t) - X_1^+(t), t \geq 0$. By Lemma 3.14, for $i = 1, 2$, Z_i satisfies the equation (3.24) with K replaced by K_i , and hence ΔZ satisfies the following nonhomogeneous transport equation:

$$\Delta Z(t, r) = \int_0^t \partial_r \Delta Z(s, r) ds + \overline{G}(r) \Delta K(t), \quad t, r \geq 0,$$

with $\Delta Z(0, \cdot) \equiv 0$. By Definition 2.6, both Z_1, Z_2 and hence ΔZ satisfy properties 1 and 2 of Lemma 3.16. Therefore, by (3.29) of Lemma 3.16, with F replaced by ΔK , we have

$$(3.30) \quad \Delta Z(t, r) = \Gamma_t \Delta K(r), \quad t, r \geq 0.$$

Since ΔK is continuous almost surely, an application of Lemma 3.6.a shows that for every $t \geq 0$, $\Delta Z(t, \cdot)$ is continuously differentiable on $[0, \infty)$, with continuous derivative

$$\partial_r \Delta Z(s, r) = -g(r) \Delta K(s) - \int_0^s \Delta K(u) g'(r + s - v) dv, \quad r, s \in [0, \infty).$$

In particular, the function $\delta(s) \doteq \partial_r \Delta Z(s, 0)$ is well defined on $[0, \infty)$ and satisfies

$$(3.31) \quad \delta(s) = -g(0) \Delta K(s) - \int_0^s \Delta K(u) g'(s - v) dv.$$

Also, recalling the constants H and H_2 from Assumptions I.b and I.c, and the fact that $\int_0^\infty \overline{G}(x) = 1$ by Assumption I.a and Remark 2.1, we see that

$$(3.32) \quad |\partial_r \Delta Z(s, r)| \leq (H + H_2) \|\Delta K\|_t, \quad \forall s \in [0, t], r \geq 0.$$

On the other hand, for $i = 1, 2$, X_i satisfies (2.21) with Z replaced by Z_i . Therefore,

$$(3.33) \quad \Delta X(t) = \int_0^t \partial_r \Delta Z(s, 0) ds = \int_0^t \delta(s) ds,$$

In turn, this implies

$$(3.34) \quad |\Delta K(t)| = |X_1^+(t) - X_2^+(t)| \leq |\Delta X(t)| \leq \int_0^t |\delta(s)| ds, \quad t \in [0, \infty),$$

which, when combined with (3.31) and the bounds in Assumption I, shows that for every $T < \infty$ and $t \in [0, T]$,

$$\begin{aligned} |\delta(t)| &\leq g(0) |\Delta K(t)| + \int_0^t |g'(t - s)| |\Delta K(s)| ds \\ &\leq H \int_0^t |\delta(s)| ds + H_2 \int_0^t \left(\int_0^v |\delta(s)| ds \right) dv \\ &\leq (H + TH_2) \int_0^t |\delta(s)| ds. \end{aligned}$$

Since $\delta(0) = 0$, by Gronwall's lemma this implies $\delta(t) = 0$ for all $t \geq 0$. When combined with (3.34), this implies that $\Delta X(t) = \Delta K(t) = 0$ for all $t \geq 0$, which in turn implies $\Delta Z(t, \cdot) = 0$ due to (3.30). Thus, we have shown that $Y_1(t) = Y_2(t)$ for all $t \geq 0$, which proves the desired uniqueness. \square

3.5. *Markov Property.* Finally, we show that the family of laws $\{P^y, y \in \mathbb{Y}\}$ associated with the diffusion model of Section 3.2 is a time-homogeneous Feller Markov family. For $s \geq 0$, we define the operator Θ_s as follows: for every function F on $[0, \infty)$, the shifted function $\Theta_s F$ is defined by

$$(3.35) \quad (\Theta_s F)(t) \doteq F(s+t) - F(s), \quad t \geq 0.$$

Also, for every $f \in \mathbb{C}_b[0, \infty)$, we define the shifted convolution integral as

$$(3.36) \quad (\Theta_s \mathcal{H})_t(f) \doteq \mathcal{M}_{s+t}(\Psi_{s+t}f) - \mathcal{M}_s(\Psi_{s+t}f), \quad t \geq 0.$$

Then, the identity (3.4) shows that

$$(3.37) \quad (\Theta_s \mathcal{H})_t(\Phi_r \mathbf{1}) = \mathcal{M}_{s+t}(\Psi_{s+t+r} \mathbf{1}) - \mathcal{M}_s(\Psi_{s+t+r} \mathbf{1}).$$

LEMMA 3.18. *Suppose Assumptions I and II hold, and let Y_0 be a \mathbb{Y} -valued random element. If $Y = (X, Z)$ is the diffusion model with initial condition Y_0 , then almost surely, for every $s \geq 0$,*

$$(3.38) \quad Z(s+t, r) = Z(s, t+r) + (\Gamma_t \Theta_s K)(r) - (\Theta_s \mathcal{H})_t(\Phi_r \mathbf{1}), \quad t, r \geq 0,$$

and $(\Theta_s K, X(s+\cdot))$ solves the CMS equation associated with $(\Theta_s E, X(s), Z(s, \cdot) - (\Theta_s \mathcal{H})(\mathbf{1}))$, i.e.,

$$(3.39) \quad (\Theta_s K, X(s+\cdot)) = \Lambda(\Theta_s E, X(s), Z(s, \cdot) - (\Theta_s \mathcal{H})(\mathbf{1})).$$

PROOF. By definitions (3.16) and (3.8) of Z and Γ_t , respectively, we have

$$(3.40) \quad Z(s, t+r) = Z_0(s+t+r) - \mathcal{M}_s(\Psi_{s+t+r} \mathbf{1}) + \overline{G}(t+r)K(s) - \int_0^s K(v)g(s+t+r-v)dv.$$

Also, by definition (3.35) of $\Theta_s K$ and (3.8) of Γ_t , we have

$$\begin{aligned} (\Gamma_t \Theta_s K)(r) &= \overline{G}(r)(\Theta_s K)(t) + \int_0^t (\Theta_s K)(v)g(t+r-v)dv \\ &= \overline{G}(r)(K(s+t) - K(s)) - \int_0^t (K(s+v) - K(s))g(t+r-v)dv \\ &= \overline{G}(r)K(s+t) - \overline{G}(r)K(s) - \int_s^{t+s} K(v)g(t+s+r-v)dv - K(s)(\overline{G}(t+r) - \overline{G}(r)) \\ (3.41) \quad &= \overline{G}(r)K(s+t) - \int_0^{t+s} K(v)g(t+s+r-v)dv - \overline{G}(t+r)K(s) \\ &\quad + \int_0^s K(v)g(t+s+r-v)dv \\ (3.42) \quad &= (\Gamma_{s+t}K)(r) - (\Gamma_s K)(t+r). \end{aligned}$$

Substituting Z from (3.16) (with t and r replaced by s and $t+r$, respectively) and using equations (3.37) and (3.42), the right-hand side of (3.38) is equal to

$$\begin{aligned} &Z_0(s+t+r) - \mathcal{M}_s(\Psi_{s+t+r} \mathbf{1}) + \Gamma_s K(t+r) + (\Gamma_{s+t}K)(r) - (\Gamma_s K)(t+r) \\ &\quad - \mathcal{M}_{s+t}(\Psi_{s+t+r} \mathbf{1}) + \mathcal{M}_s(\Psi_{s+t+r} \mathbf{1}) \\ &\quad = Z_0(s+t+r) + (\Gamma_{s+t}K)(r) - \mathcal{M}_{s+t}(\Psi_{s+t+r} \mathbf{1}), \\ &\quad = Z(s+t, r), \end{aligned}$$

where the last equality uses (3.16). This proves (3.38).

To prove (3.39), subtract equation (3.18) with $t = s$ from the same equation with t replaced by $t + s$, and use (3.14), to get

$$\begin{aligned} (\Theta_s K)(t) = K(s+t) - K(s) &= E(s+t) - E(s) - X^+(s+t) + X^+(s), \\ (3.43) \qquad \qquad \qquad &= (\Theta_s E)(t) - X^+(s+t) + X^+(s). \end{aligned}$$

Additionally, by Proposition 3.11.b with t replaced by $s+t$, $X(s+t) \wedge 0 = Z(s+t, 0)$. Substituting $Z(s+t, 0)$ from (3.38), using definition (3.8) of Γ_t and the identity $\Phi_0 \mathbf{1} = \mathbf{1}$, we obtain

$$(3.44) \qquad X(s+t) \wedge 0 = Z(s, t) - (\Theta_s \mathcal{H})_t(\mathbf{1}) + (\Theta_s K)(t) - \int_0^t (\Theta_s K)(v) g(t-v) dv.$$

Equation (3.39) then follows from (3.43), (3.44) and Definition 3.8 of Λ (also see Lemma 3.7). \square

LEMMA 3.19. *Suppose Assumptions I and II hold. Then, for every $s \geq 0$, the processes $\{\Theta_s E(t); t \geq 0\}$ and $\{(\Theta_s \mathcal{H})_t(\mathbf{1}); t \geq 0\}$ are independent of \mathcal{F}_s , and have the same distribution as the processes E and $\mathcal{H}(\mathbf{1})$, respectively. Moreover, for every $s, t \geq 0$, the $\mathbb{H}^1(0, \infty)$ -valued random element $(\Theta_s \mathcal{H})_t(\Phi \mathbf{1})$ is independent of \mathcal{F}_s and has the same distribution as $\mathcal{H}_t(\Phi \mathbf{1})$.*

PROOF. Fix $s \geq 0$. By definition (3.14) of E and (3.35) of $\Theta_s E$, we have

$$\Theta_s E(t) = E(s+t) - E(s) = \sigma B(s+t) - \sigma B(s) - \beta t, \quad t \geq 0.$$

Since B has independent stationary increments and is independent of \mathcal{M} , $B(s+t) - B(s)$ is independent of \mathcal{F}_s for all $t \geq 0$, and $\{B(s+t) - B(s); t \geq 0\}$ is itself a Brownian motion. Hence, the claim for $(\Theta_s E)$ follows.

Next, define the martingale measure $\tilde{\mathcal{M}}$ as follows: $\tilde{\mathcal{M}}_t(A) \doteq \mathcal{M}_{s+t}(A) - \mathcal{M}_s(A)$ for $t \geq 0$, $A \in \mathcal{B}[0, \infty)$. Since \mathcal{M} is a white noise independent of B , $\tilde{\mathcal{M}}$ is again a white noise with the same distribution as \mathcal{M} , and independent of \mathcal{F}_s . Also, for any continuous f , using (3.36) we have

$$\begin{aligned} (\Theta_s \mathcal{H})_t(f) &= \mathcal{M}_{s+t}(\Psi_{s+t} f) - \mathcal{M}_s(\Psi_{s+t} f) \\ &= \int_s^{s+t} \int_0^\infty \frac{\overline{G}(t+s-v+x)}{\overline{G}(x)} f(x) \mathcal{M}(dv, dx) \\ &= \int_0^t \int_0^\infty \frac{\overline{G}(t-v+x)}{\overline{G}(x)} f(x) \tilde{\mathcal{M}}(dv, dx) \\ &= \tilde{\mathcal{M}}_t(\Psi_t f). \end{aligned}$$

Substituting $f = \mathbf{1}$ and $f = \Phi_r \mathbf{1}$, we conclude that the processes $(\Theta_s \mathcal{H})_t(\mathbf{1})$ and $(\Theta_s \mathcal{H})_t(\Phi \mathbf{1})$ are independent of \mathcal{F}_s , and have the same distribution as $\{\mathcal{H}_t(\mathbf{1}); t \geq 0\}$ and $\mathcal{H}_t(\Phi \mathbf{1})$, respectively. \square

PROPOSITION 3.20. *Let Assumptions I and II hold. Then $\{P^y; y \in \mathbb{Y}\}$, where P^y is the law of the diffusion model Y^y with initial condition y , is a time-homogeneous Feller Markov family.*

PROOF. The Feller property can be deduced from the proof of Theorem 5(2) in Section 9.5 of [29], but since in our case the Markov process is homogeneous, the proof is simpler and so we include it here. Let $Y^y = (X^y, Z^y)$ be the diffusion model with initial condition $y \in \mathbb{Y}$. Recall from

Proposition 3.3 that for any $s, t \geq 0$, $\mathcal{M}_{s+t}(\Psi_{s+t+}\mathbf{1})$ and $\mathcal{M}_s(\Psi_{s+t+}\mathbf{1})$ both lie in $\mathbb{H}^1(0, \infty)$ almost surely, and hence, so does $(\Theta_s \mathcal{H})_t(\Phi \mathbf{1})$, using equation (3.37). First, we claim that for every $t \geq 0$, there exists a continuous mapping $\Pi_t : \mathbb{R} \times \mathbb{H}^1(0, \infty) \times \mathbb{C}[0, \infty)^2 \times \mathbb{H}^1(0, \infty) \mapsto \mathbb{R} \times \mathbb{H}^1(0, \infty)$ such that for $s \geq 0$,

$$(3.45) \quad Y^y(s+t) = (X^y(s+t), Z^y(s+t, \cdot)) = \Pi_t(Y^y(s), \Theta_s E(\cdot), (\Theta_s \mathcal{H})_t(\mathbf{1}), (\Theta_s \mathcal{H})_t(\Phi \mathbf{1})).$$

Recall that $\mathbb{C}^0[0, \infty)$ is the space of continuous functions f with $f(0) = 0$, and for notational convenience, set $\mathbb{D} \doteq \mathbb{R} \times \mathbb{H}^1(0, \infty) \times \mathbb{C}^0[0, \infty)^2$. To see why the claim is true, first recall that the embedding from $\mathbb{H}^1(0, \infty)$ to $\mathbb{C}[0, \infty)$ and the evaluation map $f \mapsto f(t)$ from $\mathbb{C}[0, \infty)$ to \mathbb{R} are continuous (see Lemma 2.3.b for the former). Then, by the representation of $(X^y(s+\cdot), \Theta_s K(\cdot))$ in (3.39) and the continuity of the CMS mapping $\Lambda : \mathbb{C}^0[0, \infty) \times \mathbb{R} \times \mathbb{C}[0, \infty) \mapsto \mathbb{C}^0[0, \infty) \times \mathbb{C}[0, \infty)$ from Lemma 3.7, it follows that there exists a continuous mapping $F_t^1 : \mathbb{D} \mapsto \mathbb{C}^0[0, \infty) \times \mathbb{R}$ such that for every $s \geq 0$,

$$(\Theta_s K(\cdot), X^y(s+t)) = F_t^1(X^y(s), Z^y(s, \cdot), \Theta_s E(\cdot), (\Theta_s \mathcal{H})_t(\mathbf{1})).$$

Moreover, it follows from the representation of $Z^y(s+t, \cdot)$ in (3.38), the continuity of Γ_t established in Lemma 3.6.b, and the continuity of the shift mapping $f(\cdot) \mapsto f(t+\cdot)$ on $\mathbb{H}^1(0, \infty)$, that there exists a continuous function $F_t^2 : \mathbb{H}^1(0, \infty) \times \mathbb{C}^0[0, \infty) \times \mathbb{H}^1(0, \infty) \mapsto \mathbb{H}^1(0, \infty)$ such that for every $s \geq 0$,

$$Z^y(s+t, \cdot) = F_t^2(Z^y(s, \cdot), \Theta_s K(\cdot), (\Theta_s \mathcal{H})_t(\Phi \mathbf{1})).$$

The claim follows from the last two displays.

Next, we prove the Markov property of the family $\{P^y; y \in \mathbb{Y}\}$. For every $y \in \mathbb{Y}$, $\mathcal{F}_s^y = \mathcal{F}_s \vee \sigma(y) = \mathcal{F}_s$, and hence by condition 1. of Definition 2.6 and Proposition 3.15, $(X^y(s), Z^y(s, \cdot))$ is \mathcal{F}_s -measurable. Also, $\Theta_s E(\cdot)$, $(\Theta_s \mathcal{H})_t(\mathbf{1})$ and $(\Theta_s \mathcal{H})_t(\Phi \mathbf{1})$ are independent of \mathcal{F}_s by Lemma 3.19. Hence, for every bounded and measurable functional $F : \mathbb{Y} \mapsto \mathbb{R}$, using the claim (3.45) we have

$$\begin{aligned} \mathbb{E}[F(Y^y(s+t)) | \mathcal{F}_s] &= \mathbb{E}\left[F\left(\Pi_t(Y^y(s), \Theta_s E(\cdot), (\Theta_s \mathcal{H})_t(\mathbf{1}), (\Theta_s \mathcal{H})_t(\Phi \mathbf{1}))\right) \middle| \mathcal{F}_s\right] \\ &= \mathbb{E}\left[F\left(\Pi_t(Y^y(s), \Theta_s E(\cdot), (\Theta_s \mathcal{H})_t(\mathbf{1}), (\Theta_s \mathcal{H})_t(\Phi \mathbf{1}))\right) \middle| Y^y(s)\right] \\ &= \mathbb{E}[F(Y^y(s+t)) | Y^y(s)]. \end{aligned}$$

Moreover, using the simple observation that for any two independent random variables ξ_1 and ξ_2 , and any bounded measurable function f , one has $\mathbb{E}[f(\xi_1, \xi_2) | \xi_1] = Q(\xi_1)$ where $Q(a) = \mathbb{E}[f(a, \xi_2)]$ (this is immediate to see for separable functions of the form $f(a, b) = f_1(a)f_2(b)$, and can be extended to general bounded measurable functions using the linearity of conditional expectation and the monotone convergence theorem), and applying Lemma 3.19, for every $y' \in \mathbb{Y}$, we have

$$\begin{aligned} \mathbb{E}[F(Y^y(s+t)) | Y^y(s) = y'] &= \mathbb{E}\left[F\left(\Pi_t(y', \Theta_s E(\cdot), (\Theta_s \mathcal{H})_t(\mathbf{1}), (\Theta_s \mathcal{H})_t(\Phi \mathbf{1}))\right)\right] \\ &= \mathbb{E}\left[F\left(\Pi_t(y', E, \mathcal{H}(\mathbf{1}), \mathcal{H}_t(\Phi \mathbf{1}))\right)\right] \\ (3.46) \quad &= \mathbb{E}\left[F(Y^{y'}(t))\right]. \end{aligned}$$

The last two displays show that the family $\{P^y; y \in \mathbb{Y}\}$ is a time-homogeneous Markov family.

Finally, we prove the Feller property. Recall that $\{\mathcal{P}_t\}$ denotes the transition semigroup corresponding to the Markov family $\{P^y\}$, and note that for every $t \geq 0$ and bounded continuous function F , by another use of (3.46),

$$\mathcal{P}_t[F](y) = \mathbb{E}[F(Y^y)(t)] = \mathbb{E}\left[F\left(\Pi_t(y, E, \mathcal{H}(\mathbf{1}), \mathcal{H}_t(\Phi \cdot \mathbf{1}))\right)\right], \quad y \in \mathbb{Y}.$$

By the continuity of the mapping Π_t and (3.46) and an application of the bounded convergence theorem, the mapping $y \mapsto \mathcal{P}_t[F](y)$ is continuous, and hence the Markov family $\{P^y\}$ is Feller. \square

PROOF OF THEOREM 2.7. Existence of a solution follows from Proposition 3.15 while uniqueness follows from Proposition 3.17, and the Feller Markov property is proved in Proposition 3.20. \square

4. Uniqueness of the Invariant Distribution. This section is devoted to the proof of Theorem 2.8. As mentioned in the introduction, to prove uniqueness of the invariant distribution, we will adopt the so-called *asymptotic (equivalent) coupling* method, which is particularly well suited to infinite-dimensional Markov processes. In Section 4.1, we first describe this method in the generality required for our problem, following the exposition in [22]. At the end of Section 4.1, we discuss the main steps involved in applying this framework to our problem, which are carried out in Sections 4.2–4.6.

4.1. Asymptotic Coupling: The General Framework. Let \mathcal{X} be a Polish space with a compatible metric $d(\cdot, \cdot)$, equipped with the Borel σ -algebra $\mathcal{B}(\mathcal{X})$. As usual, let $\mathcal{X}^{\mathbb{R}_+}$ denote the space of \mathcal{X} -valued functions on $[0, \infty)$. We endow $\mathcal{X}^{\mathbb{R}_+}$ with the Kolmogorov σ -algebra $\mathcal{B}(\mathcal{X})^{\mathbb{R}_+}$, which is the σ -algebra generated by all cylinder sets. Let $\mathbb{M}_1(\mathcal{X}^{\mathbb{R}_+})$ and $\mathbb{M}_1(\mathcal{X}^{\mathbb{R}_+} \times \mathcal{X}^{\mathbb{R}_+})$ denote the spaces of probability measures on $(\mathcal{X}^{\mathbb{R}_+}, \mathcal{B}(\mathcal{X})^{\mathbb{R}_+})$ and $(\mathcal{X}^{\mathbb{R}_+} \times \mathcal{X}^{\mathbb{R}_+}, \mathcal{B}(\mathcal{X})^{\mathbb{R}_+} \otimes \mathcal{B}(\mathcal{X})^{\mathbb{R}_+})$. For every $m_1, m_2 \in \mathbb{M}_1(\mathcal{X}^{\mathbb{R}_+})$, recall that a coupling of m_1 and m_2 is a probability measure $\gamma \in \mathbb{M}_1(\mathcal{X}^{\mathbb{R}_+} \times \mathcal{X}^{\mathbb{R}_+})$ whose first and second marginals, respectively, are m_1 and m_2 , that is, $\Pi_{\#}^{(i)}\gamma = m_i$ for $i = 1, 2$, where, $\Pi^{(i)}$ is the i th coordinate projection map, and $\Pi_{\#}^{(i)}\gamma$ is the push-forward of the measure γ under $\Pi^{(i)}$. Define $\mathcal{C}(m_1, m_2) \subset \mathbb{M}_1(\mathcal{X}^{\mathbb{R}_+} \times \mathcal{X}^{\mathbb{R}_+})$ to be the set of couplings of m_1 and m_2 . One can relax the definition of a coupling to define the space of *absolutely continuous couplings* as follows:

$$(4.1) \quad \tilde{\mathcal{C}}(m_1, m_2) \doteq \{\gamma \in \mathbb{M}_1(\mathcal{X}^{\mathbb{R}_+} \times \mathcal{X}^{\mathbb{R}_+}); \Pi_{\#}^{(i)}\gamma \ll m_i, i = 1, 2\}.$$

If γ in $\tilde{\mathcal{C}}(m_1, m_2)$ satisfies $\Pi_{\#}^{(i)}\gamma \sim m_i$, $i = 1, 2$, γ will be referred to as an *equivalent coupling* of m_1 and m_2 . In contrast to a coupling, the corresponding marginals of an absolutely continuous (or equivalent) coupling γ need only be absolutely continuous with respect to (respy, equivalent to), and not necessarily equal to, m_1 and m_2 , respectively.

Let $\{\mathcal{P}_t\} = \{\mathcal{P}_t; t \geq 0\}$ be the transition semigroup of a Markov kernel on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. For every $y \in \mathbb{Y}$, let P^y denote the distribution of the Markov process with initial value y and transition semigroup $\{\mathcal{P}_t\}$ on the path space $(\mathcal{X}^{\mathbb{R}_+}, \mathcal{B}(\mathcal{X})^{\mathbb{R}_+})$ (P^y is denoted in [22] by $\mathcal{P}_{[0, \infty)}\delta_y$). Recall that a probability measure μ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is called an invariant distribution for the semigroup $\{\mathcal{P}_t\}$ if (2.22) holds for every $t \geq 0$. Finally, let \mathcal{D} be the set of pairs of paths that meet at infinity:

$$(4.2) \quad \mathcal{D} \doteq \left\{ (x, y) \in \mathcal{X}^{\mathbb{R}_+} \times \mathcal{X}^{\mathbb{R}_+} : \lim_{t \rightarrow \infty} d(x(t), y(t)) = 0 \right\},$$

and note that $\mathcal{D} \in \mathcal{B}(\mathcal{X})^{\mathbb{R}_+} \otimes \mathcal{B}(\mathcal{X})^{\mathbb{R}_+}$.

Recall that a probability measure μ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is called an invariant distribution for the semigroup $\{\mathcal{P}_t\}$ if (2.22) holds for every $t \geq 0$.

PROPOSITION 4.1. *Assume there exists a measurable set $A \in \mathcal{B}(\mathcal{X})$ and a mapping $\Upsilon : A \times A \ni (y, \tilde{y}) \mapsto \Upsilon_{y, \tilde{y}} \in \tilde{\mathcal{C}}(P^y, P^{\tilde{y}})$ with the following properties:*

- (I) $\mu(A) > 0$ for any invariant probability measure μ of $\{\mathcal{P}_t\}$.
- (II) For every measurable set $B \in \mathcal{B}(\mathcal{X})^{\mathbb{R}_+} \otimes \mathcal{B}(\mathcal{X})^{\mathbb{R}_+}$, $(y, \tilde{y}) \mapsto \Upsilon_{y, \tilde{y}}(B)$ is measurable.
- (III) For every $y, \tilde{y} \in A$, $\Upsilon_{y, \tilde{y}}(\mathcal{D}) > 0$.

Then $\{\mathcal{P}_t\}$ has at most one invariant probability measure.

PROOF. Proposition 4.1 can be easily deduced from [22, Theorem 1.1. and Corollary 2.2]. For completeness, its proof is relegated to Appendix C. Also, see [14, Theorem 2] and [37, Lemma 8.5]. \square

REMARK 4.2. As observed in Remark 1.2 of [22], for the purpose of Proposition 4.1, in the definition (4.1) of $\tilde{\mathcal{C}}$, one can without loss of generality replace absolute continuity by (the apparently stronger condition of) equivalence. This is because if there is an absolutely continuous coupling Υ that satisfies conditions (II) and (III) of Proposition 4.1, then the measure $\frac{1}{2}(\Upsilon + P^y \otimes P^{\tilde{y}})$ is an equivalent coupling that also satisfies the same conditions. Thus, we refer to the approach to establishing uniqueness of the invariant distribution by invoking Proposition 4.1 as the *asymptotic equivalent coupling* approach.

In Sections 4.2-4.6, we apply the asymptotic equivalent coupling framework of Proposition 4.1, with $\mathcal{X} = \mathbb{Y}$, in order to establish uniqueness of the invariant distribution of the transition semigroup $\{\mathcal{P}_t\}$ associated to the Markov family $\{P^y; y \in \mathbb{Y}\}$ of the diffusion model defined in Section 3.2. Let A be the measurable subset of \mathbb{Y} defined as

$$(4.3) \quad A \doteq \{(x, z) \in \mathbb{Y} : x \geq 0\}.$$

First, in Section 4.2, for each pair $y, \tilde{y} \in A$, we construct a pair of stochastic processes $(Y^y, \tilde{Y}^{\tilde{y}})$ on a common probability space and define the mapping

$$(4.4) \quad \Upsilon_{y, \tilde{y}} : (y, \tilde{y}) \in A \times A \mapsto \mathcal{L}aw(Y^y, \tilde{Y}^{\tilde{y}}) \in \mathbb{M}_1(\mathbb{Y}^{\mathbb{R}_+} \times \mathbb{Y}^{\mathbb{R}_+}),$$

where $\mathcal{L}aw(Y^y, \tilde{Y}^{\tilde{y}})$ denotes the joint distribution of $(Y^y, \tilde{Y}^{\tilde{y}})$. The required measurability properties are proved in Section 4.3. Next, in Section 4.4, we show that $\Upsilon_{y, \tilde{y}}(\mathcal{D}) > 0$, and in Section 4.5 we show that the law of Y^y is P^y , and that the law of $\tilde{Y}^{\tilde{y}}$ is equivalent to $P^{\tilde{y}}$, thus establishing that Υ defines an asymptotic equivalent coupling. Finally, we combine these results in Section 4.6 to complete the proof of Theorem 2.8.

REMARK 4.3. It is worthwhile to clarify why, in Proposition 4.1, we formulate a continuous-time version of the asymptotic coupling theorem, rather than deduce the result by simply applying the original discrete-time version established in [22, Corollary 2.2] to the discrete skeleton of the continuous-time process (i.e., the Markov chain obtained by sampling at integer times). To show uniqueness of the invariant measure of the continuous-time Markov process, it clearly suffices to show uniqueness of the invariant measure of its discrete skeleton (that is, the Markov chain obtained by sampling the continuous process at integer times). In turn, by Corollary 2.2 of [22], for uniqueness of the invariant measure of the discrete skeleton, it suffices to verify the three conditions of that corollary, which are the natural discrete analogs of properties (I), (II) and (III) of Proposition 4.1.

Now, the continuous version of properties (II) and (III) immediately imply the discrete version because any asymptotic equivalent coupling of the continuous-time process on some set A induces a corresponding asymptotic equivalent coupling of the discrete skeleton. Thus, if $A = \mathcal{X}$ then the discrete-time version can be directly invoked because in this case the first condition of Corollary 2.2 of [22], which states that $\mu(A) > 0$ for every invariant measure μ of the discrete skeleton, is trivially satisfied. However, when A is a strict subset of \mathcal{X} then the property that $\mu(A) > 0$ for all invariant measures μ of the continuous-time process need not imply that $\mu(A) > 0$ for all invariant measures of the discrete skeleton, since the latter could in general be strictly larger. Moreover, in some situations (as turns out to be the case in our application), it may be relatively easy to show the former, but non-trivial to show the latter. In such cases, it is more convenient to directly apply the continuous-time version of the result, as formulated in Proposition 4.1.

4.2. Construction of a Candidate Coupling. Fix two initial conditions $y = (x_0, z_0)$ and $\tilde{y} = (\tilde{x}_0, \tilde{z}_0)$ in the set A defined in (4.3), and let $Y = Y^y$ be the diffusion model with initial condition y . Then, by Definition 3.9, $Y = (X, Z)$ and the associated process K satisfy (3.15) and (3.16) with (x_0, z_0) in place of (X_0, Z_0) . In particular, $(K, X) = \Lambda(E, x_0, z_0 - \mathcal{H}(\mathbf{1}))$. Now, define the (random) locally integrable function

$$(4.5) \quad R(t) \doteq z'_0(t) + \mathcal{H}_t(h) - g(0)K(t) - \int_0^t K(s)g'(t-s)ds.$$

Combining (3.7) and (3.20), the latter with $r = 0$, we have

$$(4.6) \quad \int_0^t R(s)ds = \int_0^t \partial_r Z(s, 0)ds.$$

We now construct another process \tilde{Y} , which starts from \tilde{y} , on the same probability space. Fix $\lambda > 0$. Given X defined above, it is easy to see that \mathbb{P} -almost surely, the linear integral equation

$$(4.7) \quad \tilde{X}(t) = \tilde{x}_0 - x_0 - \lambda \int_0^t \tilde{X}(s)ds + X(t) + \lambda \int_0^t X(s)ds,$$

has a unique continuous solution \tilde{X} , which has the form

$$(4.8) \quad \tilde{X}(t) = \tilde{x}_0 e^{-\lambda t} + F(t) - \lambda \int_0^t e^{-\lambda(t-s)} F(s)ds, \quad t \geq 0,$$

where $F(t) \doteq X(t) - x_0 + \lambda \int_0^t X(s)ds$. Since Y satisfies the diffusion model SPDE, by Proposition 3.15, X satisfies (2.21), which when combined with (4.6) and (4.7), shows that for $t \geq 0$,

$$(4.9) \quad \tilde{X}(t) = \tilde{x}_0 + \sigma B(t) - \beta t - \mathcal{M}_t(\mathbf{1}) - \lambda \int_0^t \tilde{X}(s)ds + \lambda \int_0^t X(s)ds + \int_0^t R(s)ds.$$

Also, for $t \geq 0$, define

$$(4.10) \quad \bar{K}(t) \doteq \sigma B(t) - \beta t - (\tilde{X}^+(t) - \tilde{x}_0^+) + \int_0^t (R(s) + \lambda X(s) - \lambda \tilde{X}(s))ds.$$

Clearly, \overline{K} is also continuous, almost surely. We now introduce a process \tilde{R} that, as shown in Corollary 4.5 below, can be characterized as the unique (almost surely locally integrable) solution of the following renewal equation:

$$(4.11) \quad \tilde{R}(t) = \tilde{z}'_0(t) - g(0)\overline{K}(t) - \int_0^t \overline{K}(s)g'(t-s)ds + \mathcal{H}_t(h) + \int_0^t g(t-s)\tilde{R}(s)ds.$$

In Proposition 4.4, we first collect some general results on solutions of the renewal equation. Part a of the proposition is used below in the proof of Corollary 4.5, part b is used in Section 4.4 to establish an asymptotic convergence property of the diffusion model, and part c is used in Section 4.3 to establish measurability properties of the candidate asymptotic coupling. Recall the definition of the convolution operator $*$ from Section 1.4.

PROPOSITION 4.4. *Suppose Assumption I.a holds.*

a. *If $f \in \mathbb{L}^1_{loc}(0, \infty)$, the renewal equation*

$$(4.12) \quad \varphi = f + g * \varphi$$

has a unique locally integrable solution φ_ .*

b. *If G has a finite second moment, the function f lies in $\mathbb{L}^2(0, \infty)$ and its integral $\mathcal{I}_f(t) \doteq \int_0^t f(s)ds, t \geq 0$, lies in $\mathbb{L}^2(0, \infty)$, then the solution φ_* of the renewal equation (4.12) also lies in $\mathbb{L}^2(0, \infty)$. Moreover, there exist constants $c_1, c_2 < \infty$ such that*

$$(4.13) \quad \|\varphi_*\|_{\mathbb{L}^2} \leq c_1\|f\|_{\mathbb{L}^2} + c_2\|\mathcal{I}_f\|_{\mathbb{L}^2}.$$

c. *If $f \in \mathbb{C}[0, \infty)$, then the solution φ_* of the renewal equation (4.12) also lies in $\mathbb{C}[0, \infty)$ and the mapping that takes f to $\varphi_* \in \mathbb{C}[0, \infty)$ is continuous.*

PROOF. While existence and uniqueness of solutions to renewal equations under various conditions have been extensively studied (see, e.g., [4, Theorem 2.4, p. 146]), the particular version stated above appears not to be readily available in the literature. Hence, we provide the proof in Appendix A. \square

COROLLARY 4.5. *There exists a unique locally integrable process \tilde{R} that satisfies equation (4.11).*

PROOF. Almost surely, due to the continuity of \overline{K} defined in (4.10), the continuity of $t \mapsto \mathcal{H}_t$ (see Remark 3.2), the fact that $\tilde{z}'_0 \in \mathbb{L}^2(0, \infty)$, and the local integrability of g' , the function

$$t \mapsto \tilde{z}'_0(t) - g(0)\overline{K}(t) - \int_0^t \overline{K}(s)g'(t-s)ds + \mathcal{H}_t(h)$$

lies in $\mathbb{L}^1_{loc}(0, \infty)$. The corollary then follows from Proposition 4.4.a. \square

Next, for $t \geq 0$, define

$$(4.14) \quad \tilde{K}(t) \doteq \overline{K}(t) - \int_0^t \tilde{R}(s)ds.$$

Substituting $\overline{K}(t)$ from above into (4.11) and simplifying terms, we obtain

$$(4.15) \quad \tilde{R}(t) = \tilde{z}'_0(t) + \mathcal{H}_t(h) - g(0)\tilde{K}(t) - \int_0^t \tilde{K}(u)g'(t-u)du,$$

which we observe is analogous to (4.5). Moreover, recall the definition (3.8) of the family of mappings $\{\Gamma_t; t \geq 0\}$, and in a fashion analogous to the definition of Z in (3.16), define

$$(4.16) \quad \tilde{Z}(t, r) \doteq \tilde{z}_0(t+r) - \mathcal{M}_t(\Psi_{t+r}\mathbf{1}) + \left(\Gamma_t \tilde{K}\right)(r), \quad t, r \geq 0.$$

Since \tilde{K} is almost surely continuous, it follows from (the proof of) Proposition 3.11.a that \tilde{Z} is also a continuous $\mathbb{H}^1(0, \infty)$ -valued process. Finally, set $\tilde{Y}(t) = \tilde{Y}^{\tilde{y}}(t) \doteq (\tilde{X}(t), \tilde{Z}(t))$, $t \geq 0$, and define $\Upsilon_{y, \tilde{y}}$ to be the joint distribution of $(Y^y, \tilde{Y}^{\tilde{y}})$.

4.3. Proof of the Measurability Property. In this section we prove the measurability condition (II) of Proposition 4.1 for the family of candidate equivalent couplings $\{\Upsilon_{y, \tilde{y}}, (y, \tilde{y}) \in \mathbb{Y} \times \mathbb{Y}\}$ that we constructed in the last section. Let $(Y^y, \tilde{Y}^{\tilde{y}})$ be the associated processes defined therein.

LEMMA 4.6. *Suppose Assumptions I and II hold, and let A be the measurable subset of \mathbb{Y} defined in (4.3). Then, for every $B \in \mathcal{B}(\mathcal{X})^{\mathbb{R}_+} \otimes \mathcal{B}(\mathcal{X})^{\mathbb{R}_+}$, the mapping $(y, \tilde{y}) \mapsto \Upsilon_{y, \tilde{y}}(B)$ from $A \times A$ to \mathbb{R} is measurable.*

PROOF. We claim that for every $t \geq 0$, almost surely, the mapping $(y, \tilde{y}) \mapsto (Y^y(t), \tilde{Y}^{\tilde{y}}(t))$ is continuous from $A \times A$ to $\mathbb{Y} \times \mathbb{Y}$. We first show that assuming the claim, the assertion of the lemma holds, and then we prove the claim.

Assuming that the claim holds, for every $k \geq 1$, $t_1, \dots, t_k \geq 0$ and functions $\varphi_1, \dots, \varphi_k \in \mathbb{C}_b(\mathbb{Y} \times \mathbb{Y}; \mathbb{R})$, by invoking the bounded convergence theorem we conclude that the mapping

$$(y, \tilde{y}) \mapsto \mathbb{E} \left[\varphi_1 \left(Y^y(t_1), \tilde{Y}^{\tilde{y}}(t_1) \right) \dots \varphi_k \left(Y^y(t_k), \tilde{Y}^{\tilde{y}}(t_k) \right) \right].$$

is continuous, and hence, measurable. The assertion in the lemma then follows from the definition of the Kolmogorov σ -algebra and a standard monotone class argument.

We now turn to the proof of the claim. Fix $t \geq 0$. We start by showing that the mapping from $y = (x_0, z_0) \in \mathbb{Y}$ to $Y^y(t)$ is continuous, almost surely. Fix $\omega \in \Omega^0$, where Ω^0 is a subset of Ω with $\mathbb{P}\{\Omega^0\} = 1$, on which Proposition 3.11 holds and the Brownian motion $\{B(t); t \geq 0\}$ has continuous paths. Recall again that $\mathbb{C}^0[0, \infty)$ denotes the space of continuous functions f on $[0, \infty)$ with $f(0) = 0$. The map from $y = (x_0, z_0) \in \mathbb{Y}$ to $(E, x_0, z_0 - \mathcal{H}(\mathbf{1})) \in \mathbb{C}^0[0, \infty) \times \mathbb{R} \times \mathbb{C}^0[0, \infty)$ is continuous due to Lemma 2.3.b. Since $(K, X) = \Lambda(E, x_0, z_0 - \mathcal{H}(\mathbf{1}))$ by definition (3.15), and the CMS mapping Λ is continuous by Lemma 3.7, it follows that the mapping $y \in \mathbb{Y} \mapsto (K, X) \in \mathbb{C}^0[0, \infty) \times \mathbb{C}[0, \infty)$ is continuous. In particular, for fixed $t > 0$, the mapping $y \mapsto X(t)$ is continuous and, by the definition (3.16) of $Z(t, \cdot)$, continuity of the translation map (Lemma 2.3.c) and continuity of the mapping Γ_t (Lemma 3.6.b), the mapping $y \mapsto Z(t, \cdot) \in \mathbb{H}^1[0, \infty)$ is also continuous. Thus, we have shown that the mapping $y \in \mathbb{Y} \mapsto Y^y(t) \in \mathbb{Y}$ is continuous for every $\omega \in \Omega$ and $t \geq 0$.

Next, we prove continuity of the mapping $(y, \tilde{y}) = ((x_0, z_0), (\tilde{x}_0, \tilde{z}_0)) \in A \times A \mapsto \tilde{Y}^{\tilde{y}}(t) \in \mathbb{Y}$. It is clear from the equation (4.8) and the form of the function F specified below (4.8) that the map $(x_0, \tilde{x}_0, X) \in \mathbb{R}^2 \times \mathbb{C}[0, \infty) \mapsto \tilde{X} \in \mathbb{C}[0, \infty)$ is continuous. Together with the continuity of $y \mapsto X$ proved above, this implies that the map $(y, \tilde{y}) \in \mathbb{Y}^2 \mapsto \tilde{X} \in \mathbb{C}[0, \infty)$ is continuous. Next, recall that

by the definition of A , if $y = (x_0, z_0) \in A$ then $z_0(0) = x_0 \wedge 0 = 0$. Using this identity, equation (4.6) and equation (3.20) with $r = 0$, we have

$$\mathcal{I}_R(t) \doteq \int_0^t R(s)ds = z_0(t) + \mathcal{M}_t(\mathbf{1}) - \mathcal{M}_t(\Psi_t \mathbf{1}) - \int_0^t K(u)g(t-u)du.$$

When combined with the continuity of the mappings $z_0 \in \mathbb{H}^1(0, \infty) \mapsto z_0 \in \mathbb{C}[0, \infty)$ (which holds by Lemma 2.3.b) and $y \mapsto K$ (established above), this implies that the mapping $(y, \tilde{y}) \in A \times A \mapsto \mathcal{I}_R \in \mathbb{C}[0, \infty)$ is continuous. In turn, due to the continuity of the mappings that take (y, \tilde{y}) to X and \tilde{X} established above, this implies that the map from $(y, \tilde{y}) \in A \times A$ to $\overline{K} \in \mathbb{C}[0, \infty)$ defined in (4.10) is also continuous. Next, observe from (4.11) that \tilde{R} satisfies the renewal equation $\tilde{R} = \tilde{F} + g * \tilde{R}$ where

$$\tilde{F}(t) \doteq \tilde{z}'_0(t) - g(0)\overline{K}(t) - \int_0^t \overline{K}(s)g'(t-s)ds + \mathcal{H}_t(h), \quad t \geq 0.$$

Therefore, $\mathcal{I}_{\tilde{R}}$ defined as $\mathcal{I}_{\tilde{R}}(t) = \int_0^t \tilde{R}(s)ds$, $t \geq 0$, satisfies the renewal equation $\mathcal{I}_{\tilde{R}} = \mathcal{I}_{\tilde{F}} + g * \mathcal{I}_{\tilde{R}}$, where, using the identity $\tilde{z}_0(0) = \tilde{x}_0 \wedge 0 = 0$ (which holds because $\tilde{y} \in A$), we obtain

$$\mathcal{I}_{\tilde{F}}(t) \doteq \int_0^t \tilde{F}(s)ds = \tilde{z}_0(t) - g * \overline{K}(t) + \int_0^t \mathcal{H}_s(h)ds, \quad t \geq 0.$$

Therefore, $\mathcal{I}_{\tilde{F}}$ lies in $\mathbb{C}^0[0, \infty)$ and the map from (y, \tilde{y}) to $\mathcal{I}_{\tilde{F}}$ is continuous. An application of Proposition 4.4.c then shows that $\mathcal{I}_{\tilde{R}}$ also lies in $\mathbb{C}[0, \infty)$, and that the map from $\mathcal{I}_{\tilde{F}}$ to $\mathcal{I}_{\tilde{R}}$ and hence, from (y, \tilde{y}) to $\mathcal{I}_{\tilde{R}}$, is continuous. Consequently, $\tilde{K} \in \mathbb{C}[0, \infty)$ defined in (4.14) is also obtained as a continuous map of (y, \tilde{y}) . Finally, by definition (4.16) of $\tilde{Z}(t, \cdot)$, continuity of the translation map (Lemma 2.3.c) and the continuity of the mapping Γ_t (Lemma 3.6.b), $\tilde{Z}(t, \cdot)$ can also be expressed as a continuous mapping of (y, \tilde{y}) . Therefore, the mapping $(y, \tilde{y}) \in A \times A \mapsto \tilde{Y}^{\tilde{y}}(t) \in \mathbb{Y}$ is also continuous for every $\omega \in \Omega^0$. This completes the proof of the claim, and hence, the proof of the lemma. \square

4.4. Asymptotic Convergence. In this section, we prove condition (III) of Proposition 4.1, that is, the asymptotic convergence of the processes $Y = Y^y$ and $\tilde{Y} = \tilde{Y}^{\tilde{y}}$ whenever their respective initial conditions $y = (x_0, z_0)$ and $\tilde{y} = (\tilde{x}_0, \tilde{z}_0)$ lie in the set A . Specifically, given the processes defined in Section 4.2, define $\Delta H \doteq H - \tilde{H}$ for $H = Y, y, Z, z_0, K, X, X^+, X^-, x_0$, and R . Our goal is to show that almost surely,

$$(4.17) \quad \Delta Y(t) \rightarrow 0 \text{ in } \mathbb{Y}, \quad \text{as } t \rightarrow \infty.$$

This will follow from (4.19) and Lemma 4.9 below.

Subtracting X from both sides of equation (4.7), we see that ΔX satisfies the integral equation

$$(4.18) \quad \Delta X(t) = \Delta x_0 - \lambda \int_0^t \Delta X(s)ds,$$

whose solution is given by

$$(4.19) \quad \Delta X(t) = \Delta x_0 e^{-\lambda t}.$$

Clearly, $\Delta X(t)$ converges to zero with probability one as $t \rightarrow \infty$.

We now turn to the proof of the asymptotic convergence of ΔZ , which consists of two main steps. In the first step (see Lemma 4.7) we use the fact that ΔR satisfies a certain renewal equation to show that it is square integrable. In the second step (Lemma 4.9) we combine the square integrability of ΔR with other estimates to show that $\Delta Z(t, \cdot)$ converges to zero in $\mathbb{H}^1(0, \infty)$. First, note that on substituting the expression for \bar{K} from (4.10) into the definition of \tilde{K} in (4.14), subtracting this from the equation for K in (3.18), and recalling the definition of E from (3.14), we obtain

$$(4.20) \quad \Delta K(t) = -\Delta X^+(t) + \Delta x_0^+ - \int_0^t \Delta R(s) ds - \lambda \int_0^t \Delta X(s) ds.$$

When combined with (4.18) and the fact that $x_0, \tilde{x}_0 \geq 0$, (4.20) further simplifies to

$$(4.21) \quad \Delta K(t) = -\Delta X^-(t) - \int_0^t \Delta R(s) ds.$$

LEMMA 4.7. *Let Assumptions I and II hold. Then $t \mapsto \Delta R(t) \in \mathbb{L}^2(0, \infty)$, almost surely. Moreover, there exist deterministic constants $\bar{C}_1, \bar{C}_2 < \infty$ such that*

$$(4.22) \quad \|\Delta R\|_{\mathbb{L}^2} \leq \bar{C}_1 \|\Delta z_0\|_{\mathbb{H}^1} + \bar{C}_2 |\Delta x_0|.$$

PROOF. Subtracting (4.15) from (4.5), we obtain

$$(4.23) \quad \Delta R(t) = \Delta z_0'(t) - g(0)\Delta K(t) - \int_0^t \Delta K(s)g'(t-s)ds.$$

Substituting ΔK from (4.21) into (4.23) and using Fubini's theorem, we see that ΔR satisfies the renewal equation

$$(4.24) \quad \Delta R = \bar{F} + g * \Delta R,$$

with

$$(4.25) \quad \bar{F}(t) \doteq \Delta z_0'(t) + g(0)\Delta X^-(t) + (\Delta X^- * g')(t), \quad t \geq 0.$$

We now claim that there exists $C < \infty$ such that

$$(4.26) \quad \max(\|\bar{F}\|_{\mathbb{L}^2}, \|\mathcal{I}_{\bar{F}}\|_{\mathbb{L}^2}) \leq \|\Delta z_0\|_{\mathbb{H}^1} + C|\Delta x_0| < \infty,$$

where $\mathcal{I}_{\bar{F}}(t) \doteq \int_0^t \bar{F}(s) ds$. We first show how the proof of the lemma follows from the claim. Indeed, since ΔR satisfies the renewal equation (4.24), G has a finite second moment by Assumption II, and \bar{F} and $\mathcal{I}_{\bar{F}}$ lie in $\mathbb{L}^2(0, \infty)$ by (4.26), Proposition 4.4.b implies that there exist deterministic constants such that

$$\|\Delta R\|_{\mathbb{L}^2} \leq c_1 \|\bar{F}\|_{\mathbb{L}^2} + c_2 \|\mathcal{I}_{\bar{F}}\|_{\mathbb{L}^2}.$$

When combined with (4.26), this implies that ΔR lies in $\mathbb{L}^2(0, \infty)$ and satisfies (4.22) with $\bar{C}_1 \doteq c_1 + c_2$ and $\bar{C}_2 \doteq (c_1 + c_2)C$. It only remains to prove the claim (4.26).

First, note that for every $t \geq 0$, using the fact that $\Delta z_0(0) = -\Delta x_0^- = 0$ because $y, \tilde{y} \in A$ implies $x_0^- = \tilde{x}_0^- = 0$, we have

$$\begin{aligned}
 \mathcal{I}_{\bar{F}}(t) &= \int_0^t \Delta z_0'(s) ds + g(0) \int_0^t \Delta X^-(s) ds + \int_0^t \Delta X^- * g'(s) ds \\
 &= \Delta z_0(t) + g(0) \int_0^t \Delta X^-(s) ds + \int_0^t \Delta X^-(s) (g(t-s) - g(0)) ds \\
 (4.27) \quad &= \Delta z_0(t) + \Delta X^- * g(t).
 \end{aligned}$$

Moreover, by Young's inequality (see, e.g., [6, Theorem 3.9.4]), Assumption I.c, the finite mean assumption on G and (4.19), we have

$$(4.28) \quad \|\Delta X^- * g'\|_{\mathbb{L}^2} \leq \|g'\|_{\mathbb{L}^1} \|\Delta X^-\|_{\mathbb{L}^2} \leq H_2 \|\Delta X^-\|_{\mathbb{L}^2} < \infty.$$

Together with (4.25), (4.27) and Minkowski's integral inequality, this implies that

$$\begin{aligned}
 \|\bar{F}\|_{\mathbb{L}^2} &\leq \|\Delta z_0\|_{\mathbb{H}^1} + (g(0) + H_2) \|\Delta X^-\|_{\mathbb{L}^2}, \\
 \|\mathcal{I}_{\bar{F}}\|_{\mathbb{L}^2} &\leq \|\Delta z_0\|_{\mathbb{H}^1} + H_2 \|\Delta X^-\|_{\mathbb{L}^2}.
 \end{aligned}$$

Since $\|\Delta X^-\|_{\mathbb{L}^2} \leq \frac{1}{\sqrt{2\lambda}} |\Delta x_0|$ due to (4.19), we conclude that (4.26) holds with $C \doteq (g(0) + H_2)/\sqrt{2\lambda}$. This completes the proof of the lemma. \square

In Lemma 4.9 below, we use (4.22) to prove the asymptotic convergence of $\Delta Z(t, \cdot)$. The proof of Lemma 4.9 makes use of the following elementary result on the convolution operator.

LEMMA 4.8. *Let $v \in \mathbb{L}^1(0, \infty)$, and suppose $w \in \mathbb{L}_{loc}^1(0, \infty)$ is bounded and $w(t) \rightarrow 0$ as $t \rightarrow \infty$. Then $\varphi \doteq v * w$ satisfies $\lim_{t \rightarrow \infty} \varphi(t) = 0$.*

PROOF. Fix $\varepsilon > 0$ and choose $t_0 < \infty$ such that $\text{ess sup}_{t \geq t_0} |w(t)| \leq \varepsilon$, where $\text{ess sup } f$ is the essential supremum of a function f . Then for $t \geq t_0$,

$$\phi(t) = \int_0^{t_0} w(s)v(t-s)ds + \int_{t_0}^t w(s)v(t-s)ds,$$

which implies that

$$|\phi(t)| \leq \|w\|_{\infty} \int_{t-t_0}^t |v(s)|ds + \varepsilon \|v\|_{\mathbb{L}^1}.$$

Sending $t \rightarrow \infty$ in the last display, and noting that $\int_{t-t_0}^t |v(s)|ds \rightarrow 0$ because $v \in \mathbb{L}^1(0, \infty)$, it follows that $\limsup_{t \rightarrow \infty} |\phi(t)| \leq \varepsilon \|v\|_{\mathbb{L}^1}$. Since $\varepsilon > 0$ is arbitrary, the lemma follows on sending $\varepsilon \downarrow 0$. \square

LEMMA 4.9. *Under Assumptions I and II, almost surely, $\Delta Z(t, \cdot) \rightarrow 0$ in $\mathbb{H}^1(0, \infty)$ as $t \rightarrow \infty$.*

PROOF. Subtracting (4.16) from (3.16), we see that

$$\Delta Z(t, r) = \Delta z_0(t+r) + (\Gamma_t \Delta K)(r).$$

Using (3.8) and (4.21) to expand the right-hand side above, we have

$$(4.29) \quad \Delta Z(t, r) = \Delta z_0(t + r) - \overline{G}(r) \Delta X^-(t) + \zeta(t, r) + \xi(t, r),$$

where $\zeta(t, r) \doteq \int_0^t \Delta X^-(s) g(t + r - s) ds$ and $\xi(t, r) \doteq - \int_0^t \Delta R(s) \overline{G}(t + r - s) ds$. To prove the lemma, it suffices to show that almost surely, the $\mathbb{H}^1(0, \infty)$ norm (as a function of r) of each of the terms on the right-hand side of (4.29) goes to zero as $t \rightarrow \infty$. For the first term, $\|\Delta z_0(t + \cdot)\|_{\mathbb{H}^1} \rightarrow 0$ as $t \rightarrow \infty$ by (2.16) of Lemma 2.3. For the second term, by (4.19) we have

$$\|\overline{G}(\cdot) \Delta X^-(t)\|_{\mathbb{H}^1} = \|\overline{G}\|_{\mathbb{H}^1} |\Delta X^-(t)| \leq \|\overline{G}\|_{\mathbb{H}^1} |\Delta x_0| e^{-\lambda t},$$

which converges to zero because $\|\overline{G}\|_{\mathbb{H}^1}$ is finite by Assumption I.b.

Next, since ΔX^- is continuous and g' is bounded and continuous by Assumption I.c, by the bounded convergence theorem, for each $t \geq 0$, $\zeta(t, \cdot)$ has a weak derivative

$$\partial_r \zeta(t, r) = \int_0^t \Delta X^-(s) g'(t + r - s) ds = \int_0^t \Delta X^-(t - s) g'(r + s) ds, \quad \text{a.e. } r \in (0, \infty).$$

Therefore, applying Hölder's inequality and Tonelli's theorem, we see that

$$\begin{aligned} \|\zeta(t, \cdot)\|_{\mathbb{L}^2}^2 &= \int_0^\infty \left(\int_0^t \Delta X^-(t - s) g(r + s) ds \right)^2 dr \\ &\leq \int_0^\infty \left(\int_0^t \Delta X^-(t - s)^2 g(r + s) ds \right) \left(\int_0^t g(r + s) ds \right) dr \\ &\leq \int_0^t \Delta X^-(t - s)^2 \int_0^\infty g(r + s) dr ds \\ &= ((\Delta X^-)^2 * \overline{G})(t), \end{aligned}$$

and, likewise, we have

$$\begin{aligned} \|\partial_r \zeta(t, \cdot)\|_{\mathbb{L}^2}^2 &\leq \int_0^\infty \left(\int_0^t \Delta X^-(t - s) |g'(r + s)| ds \right)^2 dr \\ &\leq H_2^2 \int_0^\infty \left(\int_0^t \Delta X^-(t - s) \overline{G}(r + s) ds \right)^2 dr \\ &\leq H_2^2 \int_0^t \Delta X^-(t - s)^2 \int_0^\infty \overline{G}(r + s) dr ds \\ &= H_2^2 \left((\Delta X^-)^2 * \int_0^\infty \overline{G}(r) dr \right) (t). \end{aligned}$$

Now, $(\Delta X^-)^2$ is integrable by (4.19) and both \overline{G} and $\int_0^\infty \overline{G}(s) ds$ are continuous, bounded by 1, lie in $\mathbb{L}^1(0, \infty)$ and vanish at infinity. Thus, it follows from Lemma 4.8 that as $t \rightarrow \infty$, $\|\zeta(t, \cdot)\|_{\mathbb{L}^2} \rightarrow 0$ and $\|\partial_r \zeta(t, \cdot)\|_{\mathbb{L}^2} \rightarrow 0$, and consequently, $\|\zeta(t, \cdot)\|_{\mathbb{H}^1} \rightarrow 0$. Similarly, since ΔR is continuous and \overline{G} has a continuous and bounded density g due to Assumption I.b, by the bounded convergence theorem, for each $t \geq 0$, $\xi(t, \cdot)$ has weak derivative

$$\partial_r \xi(t, r) = \int_0^t \Delta R(s) g(r + t - s) ds, \quad \text{a.e. } r \in (0, \infty).$$

An exactly analogous analysis then shows that

$$\|\xi(t, \cdot)\|_{\mathbb{L}^2}^2 \leq (\Delta R^2 * \overline{G})(t),$$

and

$$\|\partial_r \xi(t, \cdot)\|_{\mathbb{L}^2}^2 \leq H \left(\Delta R^2 * \int_{\cdot}^{\infty} \overline{G}(r) dr \right) (t).$$

Again, since ΔR^2 is integrable by Lemma 4.7 and both \overline{G} and $\int_{\cdot}^{\infty} \overline{G}(s) ds$ are integrable, bounded by 1 and vanish at infinity, another application of Lemma 4.8 shows that as $t \rightarrow \infty$, $\|\xi(t, \cdot)\|_{\mathbb{L}^2} \rightarrow 0$ and $\|\partial_r \xi(t, \cdot)\|_{\mathbb{L}^2} \rightarrow 0$, and consequently, $\|\xi(t, \cdot)\|_{\mathbb{H}^1} \rightarrow 0$. This completes the proof. \square

4.5. Marginal Distributions. In this section, we continue to use the ΔH notation introduced in Section 4.4. The main result of this section is as follows:

PROPOSITION 4.10. *If Assumptions I and II hold, then Y^y has distribution P^y , and the distribution of $\tilde{Y}^{\tilde{y}}$ on $\mathbb{Y}^{\mathbb{R}^+}$ is equivalent to $P^{\tilde{y}}$.*

PROOF. Y^y has distribution P^y by definition. For fixed $\tilde{y} \in \mathbb{Y}$, to show that the distribution of $\tilde{Y} = \tilde{Y}^{\tilde{y}}$ is equivalent to $P^{\tilde{y}}$, we first show that \tilde{Y} satisfies the same equations as Y , but with B replaced by some process \tilde{B} , and then invoke Girsanov's theorem to prove that \tilde{B} is a Brownian motion on the entire time interval $[0, \infty)$ under another probability measure $\tilde{\mathbb{P}}$ that is equivalent to \mathbb{P} . Let $\tilde{B}_t \doteq B_t - \int_0^t m(s) ds$, where m is defined by

$$(4.30) \quad m(s) \doteq -R(s) + \tilde{R}(s) - \lambda(X(s) - \tilde{X}(s)) = -\Delta R(s) - \lambda \Delta X(s),$$

and set $\tilde{E} \doteq \sigma \tilde{B}(t) - \beta t$. We will first show that

$$(4.31) \quad (\tilde{K}, \tilde{X}) = \Lambda(\tilde{E}, \tilde{x}_0, \tilde{z}_0 - \mathcal{H}_t(\mathbf{1})),$$

where Λ is the CMS mapping introduced in Definition 3.8. To prove (4.31), first note that the expression (4.9) for \tilde{X} can be rewritten as

$$(4.32) \quad \tilde{X}(t) = \tilde{x}_0 + \tilde{E}(t) - \mathcal{M}_t(\mathbf{1}) + \int_0^t \tilde{R}(s) ds.$$

By equation (4.15) for \tilde{R} , relation (3.7), and Fubini's theorem, we have

$$\begin{aligned} \int_0^t \tilde{R}(s) ds &= \int_0^t \tilde{z}'_0(s) ds + \int_0^t \mathcal{H}_s(h) ds - g(0) \int_0^t \tilde{K}(s) ds \\ &\quad - \int_0^t \left(\int_0^s \tilde{K}(v) g'(s-v) dv \right) ds \\ &= \tilde{z}_0(t) - \tilde{z}_0(0) - \mathcal{H}_t(\mathbf{1}) + \mathcal{M}_t(\mathbf{1}) - \int_0^t \tilde{K}(s) g(t-s) ds. \end{aligned}$$

Together with (4.32), this implies

$$(4.33) \quad \tilde{X}(t) = \tilde{x}_0 + \tilde{E}(t) + \tilde{V}(t) - \tilde{V}(0) - \tilde{K}(t),$$

where

$$(4.34) \quad \tilde{V}(t) \doteq \tilde{z}_0(t) - \mathcal{H}_t(\mathbf{1}) + \tilde{K}(t) - \int_0^t \tilde{K}(s)g(t-s)ds.$$

Also, by definitions (4.10) and (4.14) of \overline{K} and \tilde{K} and equation (4.30), we have

$$(4.35) \quad \tilde{K}(t) = \overline{K}(t) - \int_0^t \tilde{R}(s)ds = \tilde{E}(t) - \tilde{X}^+(t) + \tilde{x}_0^+.$$

Substituting \tilde{K} from (4.35) in (4.33) and noting from (4.34) and the fact that $(\tilde{x}_0, \tilde{z}_0) \in \mathbb{Y}$, $\tilde{V}(0) = \tilde{z}_0(0) = -\tilde{x}_0^-$, we conclude that

$$(4.36) \quad \tilde{V}(t) = \tilde{X}(t) \wedge 0 = -\tilde{X}(t)^-.$$

Comparing the equation obtained on substituting \tilde{V} from (4.36) into (4.35) with the CMS equation (3.17), and comparing (4.35) with the CMS equation (3.18), it follows that (4.31) holds. Furthermore, \tilde{Z} defined in (4.16) has the same form as the expression (3.16) for Z , but with K replaced by \tilde{K} . In summary, we have shown that \tilde{Y} is defined in the same way as the process $Y^{\tilde{y}}$ in Definition 3.9, except that B is replaced by \tilde{B} . Therefore, to complete the proof, it suffices to show that there exists a new probability measure $\tilde{\mathbb{P}}$ on (Ω, \mathcal{F}) that is equivalent to \mathbb{P} and under $\tilde{\mathbb{P}}$, \tilde{B} is a Brownian motion independent of \mathcal{M} .

Define the process $\{N_t; t \geq 0\}$ as

$$(4.37) \quad N_t \doteq \exp \left(\int_0^t m(s)dB(s) - \frac{1}{2} \int_0^t m^2(s)ds \right), \quad t \geq 0,$$

where $m = -(\Delta R + \lambda \Delta X)$ is as above. Consider the local martingale $M(t) \doteq \int_0^t m(s)dB_s$, with quadratic variation $\langle M \rangle_t = \int_0^t m^2(s)ds$, $t \geq 0$. Note that $\langle M \rangle_\infty = \|m\|_{\mathbb{L}^2}^2$, and by (4.30), (4.22) and (4.19), there exist constants $C, \bar{C}_1, \bar{C}_2 < \infty$ such that

$$\|m\|_{\mathbb{L}^2}^2 \leq 2\|\Delta R\|_{\mathbb{L}^2}^2 + 2\lambda^2\|\Delta X\|_{\mathbb{L}^2}^2 \leq 4\bar{C}_1^2\|z_0\|_{\mathbb{H}^1}^2 + (4\bar{C}_2^2 + \lambda)|\Delta x_0|^2 \doteq C < \infty.$$

Therefore, $\mathbb{E}[\exp(\langle M \rangle_\infty/2)] < \infty$, which implies that N is a uniformly integrable exponential martingale (see, e.g., [30, Section 3]). Then, by Doob's convergence theorem, N_t converges almost surely as $t \rightarrow \infty$ to an integrable random variable N_∞ . The inequality $\langle M \rangle_\infty < \infty$ also implies that M is itself a uniformly integrable martingale. Therefore, by another application of Doob's convergence theorem, almost surely, $M(t)$ has a finite limit as $t \rightarrow \infty$, which ensures that N_∞ is almost surely positive. Define a probability measure $\tilde{\mathbb{P}}$ on (Ω, \mathcal{F}) by:

$$(4.38) \quad \tilde{\mathbb{P}}(A) = \mathbb{E}[\mathbb{1}_A N_\infty], \quad A \in \mathcal{F},$$

where \mathbb{E} denotes expectation with respect to \mathbb{P} . Since N_∞ is almost surely positive, $\tilde{\mathbb{P}}$ is equivalent to \mathbb{P} . Also, by Girsanov's theorem (see, e.g., [43, Theorem (38.5), Chapter IV]), $\{\tilde{B}(t) = B(t) - \int_0^t m(s)ds, t \geq 0\}$ is a Brownian motion under $\tilde{\mathbb{P}}$. Moreover, recall that under \mathbb{P} , for every $A_1, A_2 \in \mathcal{B}[0, \infty)$, $\mathcal{M}(A_i)$, $i = 1, 2$, is a martingale independent of B , $\langle \mathcal{M}(A_i), B \rangle \equiv 0$. Thus, by [27, Proposition 5.4, Chapter 3] (note that since N is a martingale, for every $T > 0$ and $A \in \mathcal{F}_T$, $\tilde{\mathbb{P}}(A) = \mathbb{E}[\mathbb{1}_A N_T]$, and hence, our definition of $\tilde{\mathbb{P}}$ is compatible with its definition (5.4) in [27, Chapter 3]) under $\tilde{\mathbb{P}}$, $\mathcal{M}(A_i)$ is a martingale, $\langle \mathcal{M}(A_1), \mathcal{M}(A_2) \rangle_t = t \int_0^\infty \mathbb{1}_{A_1 \cap A_2}(x) g(x)dx$ and $\langle \tilde{B}, \mathcal{M}(A_i) \rangle \equiv 0$ for $i = 1, 2$. Therefore, under $\tilde{\mathbb{P}}$, \mathcal{M} is a martingale measure with covariance function given in (2.2) and is independent of \tilde{B} . This completes the proof. \square

4.6. Proof of Theorem 2.8.

PROOF OF THEOREM 2.8. Proposition 3.20 shows that the diffusion model Y is a time-homogeneous Feller Markov process on the Polish space \mathbb{Y} . Let $\mathcal{P} = \{\mathcal{P}_t, t \geq 0\}$ be the transition semigroup associated to Y . In order to show that $\{\mathcal{P}_t\}$ has at most one invariant distribution, it suffices to show that the candidate coupling $\{\Upsilon_{y,\tilde{y}}, (y, \tilde{y}) \in A^2\}$ constructed in Section 4.2 satisfies the conditions of Proposition 4.1. Recall that $A = \{(x, z) \in \mathbb{Y} ; x \geq 0\}$. For every $y, \tilde{y} \in A$, it follows from Lemma 4.6 that the mapping $(y, \tilde{y}) \mapsto \Upsilon_{y,\tilde{y}}(B)$ is measurable for every $B \in \mathcal{B}(\mathcal{X})^{\mathbb{R}+} \otimes \mathcal{B}(\mathcal{X})^{\mathbb{R}+}$ and from Proposition 4.10 that $\Upsilon_{y,\tilde{y}} \in \tilde{\mathcal{C}}(P^y, P^{\tilde{y}})$. Moreover, (4.17) follows from (4.19) and Lemma 4.9, and hence, $\Upsilon_{y,\tilde{y}}(\mathcal{D}) = 1$. So, to complete the proof of the theorem, it suffices to show that the subset A satisfies the first condition of Proposition 4.1.

Let μ be an invariant distribution of $\{\mathcal{P}_t\}$. Assume to the contrary that $\mu(A) = 0$. Let Y_0 be a \mathbb{Y} -valued random element distributed as μ , and let $Y = (X, Z)$ be the diffusion model with initial condition Y_0 . Since μ is invariant for $\{\mathcal{P}_t\}$, for every $t \geq 0$, $Y(t)$ is also distributed as μ , and therefore, $\mathbb{P}\{Y(t) \in A\} = \mathbb{P}\{X(t) \geq 0\} = 0$. Equivalently, we have $\mathbb{P}\{X(t) < 0\} = 1$ for all $t \geq 0$. Since X has continuous sample paths almost surely, this implies that

$$(4.39) \quad \mathbb{P}\{X(t) \leq 0 \text{ for every } t \geq 0\} = 1.$$

By (3.17), (3.18) and (4.39), almost surely, for every $t \geq 0$ we have $K(t) = \sigma B(t) - \beta t - X^+(t) + X^+(0) = \sigma B(t) - \beta t$, and

$$\begin{aligned} X(t) = X(t) \wedge 0 &= Z_0(t) + \sigma B(t) - \beta t - \mathcal{H}_t(\mathbf{1}) - \int_0^t g(t-s)(\sigma B(s) - \beta s)ds \\ &= Z_0(t) - \beta t + \beta \int_0^t sg(t-s)ds + \sigma \int_0^t \overline{G}(t-s)dB_s - \mathcal{H}_t(\mathbf{1}). \end{aligned}$$

For every $t > 0$, $\sigma \int_0^t \overline{G}(t-s)dB_s$ and $-\mathcal{H}_t(\mathbf{1})$ are two independent finite variance Gaussian random variables that are independent of $Z_0(\cdot)$. Therefore, $X(t)$ is the sum of the random variable $Z_0(t)$, an independent zero-mean Gaussian random variable with finite covariance, and a finite constant $c_t \doteq \beta t - \beta \int_0^t sg(t-s)ds$, which therefore satisfies $\mathbb{P}\{X(t) > 0\} > 0$. This contradicts (4.39), and hence, $\mu(A) > 0$, and the proof is complete. \square

5. Proofs of Preliminary Results. We now provide the proofs of various results stated in Section 3.1.

5.1. *Martingale Measure Integrals.* In this section, we prove the regularity properties of stochastic integrals stated in Section 3.1.1.

5.1.1. Continuity of martingale measures.

PROOF OF LEMMA 3.1. Fix $T \in (0, \infty)$, $0 \leq s \leq t \leq T$ and $0 \leq r, u \leq T$. Recalling from (2.2) the quadratic variation process of integrals with respect to \mathcal{M} , and using the Burkholder-Davis-Gundy inequality for martingales (see e.g. [45, Theorem 7.11]), there exists a constant $\hat{c}_1 < \infty$ such that for every bounded measurable function φ on $[0, \infty) \times [0, \infty)$,

$$(5.1) \quad \mathbb{E} \left[|\mathcal{M}_t(\varphi) - \mathcal{M}_s(\varphi)|^6 \right] \leq \hat{c}_1 \left(\int_0^t \int_0^\infty \varphi^2(x, v)g(x)dx dv \right)^3.$$

Let $\hat{c}_2 < \infty$ be such that $(a+b)^6 \leq \hat{c}_2(a^6 + b^6)$, and set $c \doteq \hat{c}_1 \hat{c}_2 < \infty$. Then, by (3.2) and the bound (5.1), we have

$$\begin{aligned} \mathbb{E} [|\mathcal{M}_t(\Phi_r \mathbf{1}) - \mathcal{M}_s(\Phi_u \mathbf{1})|^6] &\leq c \mathbb{E} [|\mathcal{M}_t(\Phi_r \mathbf{1}) - \mathcal{M}_s(\Phi_r \mathbf{1})|^6] + c \mathbb{E} [|\mathcal{M}_s(\Phi_r \mathbf{1}) - \mathcal{M}_s(\Phi_u \mathbf{1})|^6] \\ &\leq c \left(\int_s^t \int_0^\infty \frac{\overline{G}(x+r)^2}{\overline{G}(x)^2} g(x) dx dv \right)^3 \\ &\quad + c \left(\int_0^s \int_0^\infty \frac{(\overline{G}(x+r) - \overline{G}(x+u))^2}{\overline{G}(x)^2} g(x) dx dv \right)^3. \end{aligned}$$

where $c \doteq \hat{c}_1 \hat{c}_2 < \infty$. Then, by Assumption I.b and the mean value theorem, there exists $r^* \in [r, u]$ such that

$$\begin{aligned} \mathbb{E} [|\mathcal{M}_t(\Phi_r \mathbf{1}) - \mathcal{M}_s(\Phi_u \mathbf{1})|^6] &\leq c|t-s|^3 + cT^3 \left(\int_0^\infty \frac{g(x+r^*)^2}{\overline{G}(x)^2} g(x) dx \right)^3 |r-u|^6 \\ &\leq c|t-s|^3 + cT^3 H^6 (2T)^3 |r-u|^3. \end{aligned}$$

Note that the inequality $|r-u| \leq 2T$ is used in the second line, which holds because $r, u \in [0, T]$. Therefore, by the Kolmogorov-Centsov theorem (see, e.g., [42, Theorem I.25.2], with $n=2$, $\alpha=6$ and $\epsilon=1$), the random field $\{\mathcal{M}_t(\Phi_r \mathbf{1}); t, r \geq 0\}$ has a continuous modification.

In a similar fashion, by definition (3.1) and another application of the bound (5.1), there exists $r^* \in [t+r, s+u]$ such that

$$\begin{aligned} \mathbb{E} [|\mathcal{M}_t(\Psi_{t+r} \mathbf{1}) - \mathcal{M}_s(\Psi_{s+u} \mathbf{1})|^6] &\leq c \left(\int_s^t \int_0^\infty \frac{\overline{G}(t+x+r-v)^2}{\overline{G}(x)^2} g(x) dx dv \right)^3 \\ &\quad + c \left(\int_0^s \int_0^\infty \frac{(\overline{G}(t+x+r-v) - \overline{G}(s+x+u-v))^2}{\overline{G}(x)^2} g(x) dx dv \right)^3 \\ &\leq c|t-s|^3 + c \left(\int_0^s \int_0^\infty \frac{g(x+r^*-v)^2}{\overline{G}(x)^2} g(x) dx dv \right)^3 (|t-s| + |r-u|)^6 \\ &\leq c|t-s|^3 + \hat{c}_2 cT^3 H^6 (2T)^3 (|t-s|^3 + |r-u|^3) \\ &\leq \tilde{c}(|t-s|^3 + |r-u|^3), \end{aligned}$$

for some $\tilde{c} < \infty$. Again, by the Kolmogorov-Centsov theorem, $\{\mathcal{M}_t(\Psi_{t+r} \mathbf{1}); t, r \geq 0\}$ has a continuous modification. Similarly, using Assumption I.c and (3.1), we obtain

$$\begin{aligned} \mathbb{E} [|\mathcal{M}_t(\Psi_{t+r} h) - \mathcal{M}_s(\Psi_{s+u} h)|^6] &\leq c|t-s|^3 + c3T^3 H_2^6 (2T)^3 |t+r-s-u|^3 \\ &\leq \tilde{c}(|t-s|^3 + |r-u|^3), \end{aligned}$$

for some $c, \tilde{c} < \infty$. Thus, $\{\mathcal{M}_t(\Psi_{t+r} h); t, r \geq 0\}$ has a continuous modification. \square

5.1.2. More Sample Path Properties. The goal of this section is to prove Proposition 3.3. We break the argument into several steps. We start by establishing properties of the random element $\mathcal{M}_t(\Psi_{t+} \mathbf{1})$ for fixed $t \geq 0$.

LEMMA 5.1. *If Assumption I.b holds, then the following two properties are satisfied:*

a. *For every $t \geq 0$, almost surely,*

$$(5.2) \quad \mathcal{M}_t(\Psi_{t+r}\mathbf{1}) = \mathcal{H}_t(\mathbf{1}) - \int_0^r \mathcal{M}_t(\Psi_{t+u}h)du, \quad \forall r \geq 0;$$

b. *If, in addition, Assumption II holds, then for every $t \geq 0$, the random functions $r \mapsto \mathcal{M}_t(\Psi_{t+r}\mathbf{1})$ and $r \mapsto \mathcal{M}_t(\Psi_{t+r}h)$ lie in $\mathbb{L}^2(0, \infty)$, almost surely.*

PROOF. We start by proving property a, which is similar in spirit to, but not a direct consequence of, Lemma E.1 of [29]. For fixed $t > 0$, by Assumption I.b, the function

$$(\Psi_{t+u}h)(x, s) = \frac{g(x+t+u-s)}{\overline{G}(x)},$$

is measurable in (u, x, s) , bounded and satisfies

$$\int_0^r (\Psi_{t+u}h)(x, s)du = \frac{\overline{G}(x+t-s)}{\overline{G}(x)} - \frac{\overline{G}(x+t+r-s)}{\overline{G}(x)} = \Psi_t\mathbf{1}(x, s) - \Psi_{t+r}\mathbf{1}(x, s).$$

Since \mathcal{M} is an orthogonal, and therefore a worthy, martingale measure, by the stochastic Fubini theorem for martingale measures (see e.g. [45, Theorem 2.6]) we have

$$\int_0^r \mathcal{M}_t(\Psi_{t+u}h)du = \mathcal{M}_t\left(\int_0^r \Psi_{t+u}h du\right) = \mathcal{M}_t(\Psi_t\mathbf{1}) - \mathcal{M}_t(\Psi_{t+r}\mathbf{1}),$$

Since $\mathcal{H}_t(\mathbf{1}) = \mathcal{M}_t(\Psi_t\mathbf{1})$, equation (5.2) follows from the last display.

Now we turn to the proof of property b. For every $t, r \geq 0$, (2.4), (3.1) and Assumption I.b imply that

$$\begin{aligned} \mathbb{E}[\mathcal{M}_t(\Psi_{t+r}\mathbf{1})^2] &= \int_0^t \int_0^\infty \frac{\overline{G}(x+t+r-s)^2}{\overline{G}(x)^2} g(x) dx ds \\ &\leq t \int_0^\infty \frac{\overline{G}(x+r)^2}{\overline{G}(x)^2} g(x) dx \\ &\leq tH \int_r^\infty \overline{G}(x) dx. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mathbb{E}[\mathcal{M}(\Psi_{t+r}h)^2] &= \int_0^t \int_0^\infty \frac{g(x+t+r-s)^2}{\overline{G}(x)^2} g(x) dx ds \\ &\leq H^2 \int_0^t \int_0^\infty \frac{\overline{G}(x+t+r-s)^2}{\overline{G}(x)^2} g(x) dx ds \\ &\leq tH^3 \int_r^\infty \overline{G}(x) dx. \end{aligned}$$

Therefore, Fubini's theorem and the last two inequalities together show that

$$\max\left(\mathbb{E}\left[\int_0^\infty \mathcal{M}_t(\Psi_{t+r}\mathbf{1})^2 dr\right], \mathbb{E}\left[\int_0^\infty \mathcal{M}_t(\Psi_{t+r}h)^2 dr\right]\right) \leq t(H^3 \vee H) \int_0^\infty \int_r^\infty \overline{G}(x) dx dr,$$

which is finite by Assumption II (see Remark 2.2). Therefore, the \mathbb{L}^2 norms of both $\mathcal{M}_t(\Psi_{t+\cdot}\mathbf{1})$ and $\mathcal{M}_t(\Psi_{t+\cdot}h)$ are finite in expectation, and hence finite almost surely. \square

COROLLARY 5.2. *Suppose Assumptions I.b and II hold. Then for every $t \geq 0$, almost surely, the function $\mathcal{M}_t(\Psi_{t+}\mathbf{1})$ lies in $\mathbb{H}^1(0, \infty)$ and has weak derivative $-\mathcal{M}_t(\Psi_{t+}.h)$.*

PROOF. Fix $t \geq 0$. It follows from part a of Lemma 5.1 that almost surely, the function $r \mapsto \mathcal{M}_t(\Psi_{t+r}\mathbf{1})$ is (locally) absolutely continuous with density (and hence, weak derivative) $-\mathcal{M}_t(\Psi_{t+}.h)$. Moreover, part b of the same lemma shows that both the function $\mathcal{M}_t(\Psi_{t+}\mathbf{1})$ and its weak derivative, $-\mathcal{M}_t(\Psi_{t+}.h)$, lie in $\mathbb{L}^2(0, \infty)$, almost surely. \square

Corollary 5.2 shows that $\{\mathcal{M}_t(\Psi_{t+}\mathbf{1}); t \geq 0\}$ is an $\mathbb{H}^1(0, \infty)$ -valued stochastic process. Next, we show that this process has a continuous modification.

LEMMA 5.3. *Let Assumptions I and II hold. Then the $\mathbb{H}^1(0, \infty)$ -valued process $\{\mathcal{M}_t(\Psi_{t+}\mathbf{1}); t \geq 0\}$ has a continuous modification.*

PROOF. Choose $T \geq 0$ and $0 \leq s \leq t \leq T$, and define

$$(5.3) \quad \zeta_{s,t}(r) \doteq \mathcal{M}_t(\Psi_{t+r}\mathbf{1}) - \mathcal{M}_s(\Psi_{s+r}\mathbf{1}), \quad r \in (0, \infty).$$

Then for every $r \geq 0$,

$$\begin{aligned} \zeta_{s,t}(r) &= \iint_{[0, \infty) \times [0, t]} \Psi_{t+r}\mathbf{1}(x, v) \mathcal{M}(dx, dv) - \iint_{[0, \infty) \times [0, s]} \Psi_{s+r}\mathbf{1}(x, v) \mathcal{M}(dx, dv) \\ &= \iint_{[0, \infty) \times [0, s]} (\Psi_{t+r}\mathbf{1}(x, v) - \Psi_{s+r}\mathbf{1}(x, v)) \mathcal{M}(dx, dv) \\ &\quad + \iint_{[0, \infty) \times (s, t]} \Psi_{t+r}\mathbf{1}(x, v) \mathcal{M}(dx, dv). \end{aligned}$$

Since $\Psi_{t+}\mathbf{1}$ and $\Psi_{s+}\mathbf{1}$ are deterministic functions and \mathcal{M} has independent increments, this shows that $\{\zeta_{s,t}(r); r \geq 0\}$ is the sum of two independent Gaussian processes. Therefore, $\{\zeta_{s,t}(r); r \geq 0\}$ is a Gaussian process with covariance function $\sigma(r, u) = \sigma_1(r, u) + \sigma_2(r, u)$, where for $r, u \geq 0$,

$$\begin{aligned} \sigma_1(r, u) &\doteq \mathbb{E}[\mathcal{M}_s(\Psi_{t+r}\mathbf{1} - \Psi_{s+r}\mathbf{1})\mathcal{M}_s(\Psi_{t+u}\mathbf{1} - \Psi_{s+u}\mathbf{1})], \\ \sigma_2(r, u) &\doteq \mathbb{E} \left[\iint_{[0, \infty) \times (s, t]} \Psi_{t+r}\mathbf{1}(x, v) \mathcal{M}(dx, dv) \iint_{[0, \infty) \times (s, t]} \Psi_{t+u}\mathbf{1}(x, v) \mathcal{M}(dx, dv) \right]. \end{aligned}$$

Using the fact that for $0 \leq s, t < \infty$ and $a \geq 0$,

$$(\Psi_{t+a}\mathbf{1} - \Psi_{s+a}\mathbf{1})(x, v) = \frac{\overline{G}(t+a+x-v) - \overline{G}(s+a+x-v)}{\overline{G}(x)},$$

by the mean value theorem for \overline{G} , (2.4), Assumption I.b, and the monotonicity of \overline{G} , we have for some $t_1, t_2 \in (s, t)$,

$$\begin{aligned} \sigma_1(r, u) &= |t-s|^2 \int_0^s \int_0^\infty \frac{g(t_1+r+x-v)g(t_2+u+x-v)}{\overline{G}(x)^2} g(x) dx dv \\ &\leq TH^3 |t-s|^2 \int_0^\infty \frac{\overline{G}(x+r)\overline{G}(x+u)}{\overline{G}(x)} dx, \end{aligned}$$

and an analogous calculation shows that

$$\begin{aligned}\sigma_2(r, u) &= \int_s^t \int_0^\infty \frac{\overline{G}(t+r+x-v)\overline{G}(t+u+x-v)}{\overline{G}(x)^2} g(x) dx dv \\ &\leq H|t-s| \int_0^\infty \frac{\overline{G}(x+r)\overline{G}(x+u)}{\overline{G}(x)} dx.\end{aligned}$$

Setting $C_T \doteq (1 + T^2 H^2)H$, this implies that for every $r, u \geq 0$,

$$(5.4) \quad \sigma(r, u) \leq C_T |t-s| \int_0^\infty \frac{\overline{G}(x+r)\overline{G}(x+u)}{\overline{G}(x)} dx.$$

Recall that given a pair of jointly Gaussian random variables (ξ_1, ξ_2) with covariance matrix $\Sigma = [\sigma(i, j)]_{i,j=1,2}$, $\mathbb{E} [\xi_1^2 \xi_2^2] = \sigma(1, 1)\sigma(2, 2) + \sigma(1, 2)^2$. Applying this identity with $\xi_1 = \zeta_{s,t}(r)$ and $\xi_2 = \zeta_{s,t}(u)$, and using (5.4) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}(5.5) \quad \mathbb{E} [\zeta_{s,t}(r)^2 \zeta_{s,t}(u)^2] &\leq C_T^2 |t-s|^2 \left(\int_0^\infty \frac{\overline{G}(x+r)^2}{\overline{G}(x)} dx \right) \left(\int_0^\infty \frac{\overline{G}(x+u)^2}{\overline{G}(x)} dx \right) \\ &\quad + C_T^2 |t-s|^2 \left(\int_0^\infty \frac{\overline{G}(x+r)\overline{G}(x+u)}{\overline{G}(x)} dx \right)^2 \\ &\leq 2C_T^2 |t-s|^2 \left(\int_r^\infty \overline{G}(x) dx \right) \left(\int_u^\infty \overline{G}(x) dx \right).\end{aligned}$$

Then, by Tonelli's theorem,

$$\begin{aligned}\mathbb{E} [\|\zeta_{s,t}(\cdot)\|_{\mathbb{L}^2}^4] &= \mathbb{E} \left[\left(\int_0^\infty \zeta_{s,t}^2(r) dr \right)^2 \right] \\ &= \mathbb{E} \left[\int_0^\infty \int_0^\infty \zeta_{s,t}^2(r) \zeta_{s,t}^2(u) dr du \right] \\ &= \int_0^\infty \int_0^\infty \mathbb{E} [\zeta_{s,t}^2(r) \zeta_{s,t}^2(u)] dr du.\end{aligned}$$

The bound (5.5) then implies that

$$\mathbb{E} [\|\zeta_{s,t}(\cdot)\|_{\mathbb{L}^2}^4] \leq 2C_T^2 |t-s|^2 \left(\int_0^\infty \int_r^\infty \overline{G}(x) dx dr \right)^2 \leq \tilde{C}_T |t-s|^2$$

for $\tilde{C}_T \doteq 2C_T^2 \left(\int_0^\infty \int_r^\infty \overline{G}(x) dx dr \right)^2$, which is finite by Assumption II (see Remark 2.2). Substituting the definition (5.3) of $\zeta_{s,t}$ into the last inequality, it follows that

$$(5.6) \quad \mathbb{E} [\|\mathcal{M}_t(\Psi_{t+} \cdot \mathbf{1}) - \mathcal{M}_s(\Psi_{s+} \cdot \mathbf{1})\|_{\mathbb{L}^2}^4] \leq \tilde{C}_T |t-s|^2.$$

Now, note that by (5.3) and Corollary 5.2, almost surely, $\zeta_{s,t}(\cdot)$ has weak derivative

$$\zeta'_{s,t}(r) \doteq -\mathcal{M}_t(\Psi_{t+r} h) + \mathcal{M}_s(\Psi_{s+r} h).$$

Using estimates analogous to those used above, one can show that

$$(5.7) \quad \mathbb{E} [\|\mathcal{M}_t(\Psi_{t+} h) - \mathcal{M}_s(\Psi_{s+} h)\|_{\mathbb{L}^2}^4] \leq C'_T |t-s|^2,$$

where now

$$C'_T = 2 \left(H^2 + 2T^2 H H_2^2 \right)^2 \left(\int_0^\infty \int_r^\infty \overline{G}(x) dx dr \right)^2,$$

which is finite by Assumptions I and II. Combining (5.6) and (5.7), for every $s, t \leq T$, we have

$$\mathbb{E} [\|\mathcal{M}_t(\Psi_{t+}h) - \mathcal{M}_s(\Psi_{s+}h)\|_{\mathbb{H}^1}^4] \leq \tilde{C}_T |t - s|^2,$$

for a finite \tilde{C}_T that depends only on T . The existence of an $\mathbb{H}^1(0, \infty)$ -continuous modification of $\mathcal{M}_t(\Psi_{t+}h)$ then follows from a version of Kolmogorov's continuity criterion for stochastic processes taking values in general Polish spaces (see, e.g., Lemma 2.1 in [44]). \square

PROOF OF PROPOSITION 3.3. Proposition 3.3 follows from Corollary 5.2 and Lemma 5.3. \square

5.1.3. *A Fubini-Type Lemma.* Here, we present the proof of Lemma 3.5. We first prove a weaker version of this lemma.

LEMMA 5.4. *Suppose Assumption I is satisfied. Then, for every $r \geq 0$, almost surely,*

$$(5.8) \quad \mathcal{M}_t(\Psi_{t+r}\mathbf{1}) = \mathcal{M}_t(\Phi_r\mathbf{1}) - \int_0^t \mathcal{M}_s(\Psi_{s+r}h) ds, \quad \forall t \geq 0.$$

PROOF. As shown below, this will follow as a corollary of Lemma E.1 in [29], which is applicable here since Assumptions 2 and 4 in [29] and the continuity of g can be deduced from Assumptions I.a and I.b of the current article. (Note that although the statement of Lemma E.1 in [29] imposes Assumptions 1-4 of [29], Assumptions 1 and 3 of [29] are not used in the proof of that lemma.)

Following definition (8.27) in [29], for $t \geq 0$, define the operator $\Xi_t : \mathbb{C}_b([0, \infty) \times [0, t]) \mapsto \mathbb{C}_b([0, \infty) \times [0, t])$ by

$$(5.9) \quad (\Xi_t \varphi)(x, u) \doteq \int_u^t (\Psi_s \varphi(\cdot, s))(x, u) ds = \int_u^t \varphi(x + s - u, s) \frac{\overline{G}(x + s - u)}{\overline{G}(x)} ds.$$

Then (E.1) of Lemma E.1 of [29] shows that given any $\tilde{\varphi} \in \mathbb{C}_b([0, \infty) \times [0, t])$, almost surely,

$$(5.10) \quad \mathcal{M}_t(\Xi_t \tilde{\varphi}) = \int_0^t \mathcal{H}_r(\tilde{\varphi}(\cdot, r)) dr, \quad t \geq 0.$$

Now, fix $r \geq 0$, and define

$$\varphi_r(x, u) \doteq \Phi_r h(x) = \frac{g(x + r)}{\overline{G}(x)}, \quad (x, u) \in [0, \infty) \times [0, t].$$

Note that for every $t \geq 0$, $\varphi_r \in \mathbb{C}_b([0, \infty) \times [0, t])$ due to Assumption I.a and I.b, and

$$(\Xi_t \varphi_r)(x, u) = \int_u^t \frac{g(x + s + r - u)}{\overline{G}(x)} ds = \frac{\overline{G}(x + r)}{\overline{G}(x)} - \frac{\overline{G}(x + t + r - u)}{\overline{G}(x)} = \Phi_r \mathbf{1}(x) - \Psi_{t+r} \mathbf{1}(x, u).$$

Applying (5.10) with $\tilde{\varphi} = \varphi_r = \Phi_r h$, and substituting the last identity, we see that almost surely,

$$\mathcal{M}_t(\Phi_r \mathbf{1}) - \mathcal{M}_t(\Psi_{t+r} \mathbf{1}) = \int_0^t \mathcal{H}_s(\Phi_r h) ds, \quad \forall t \geq 0.$$

Since equation (3.5) for \mathcal{H}_t and property (3.4) of the operators $\{\Psi_t; t \geq 0\}$ imply that

$$\mathcal{H}_s(\Phi_r h) = \mathcal{M}_s(\Psi_s \Phi_r h) = \mathcal{M}_s(\Psi_{s+r} h) \quad \forall s, r \geq 0,$$

this concludes the proof of the lemma. \square

We now prove Lemma 3.5.

PROOF OF LEMMA 3.5. It follows from Lemma 3.1 and Remark 3.2 that both sides of (5.8) are jointly continuous in (t, r) . Therefore, Lemma 5.4 can be strengthened to obtain Lemma 3.5, namely, almost surely, for every $r, t \geq 0$, the equality in (5.8) is satisfied. \square

5.2. *The Auxiliary Mapping Γ and the Transport Equation.* In Section 5.2.1 we prove Lemma 3.6, which characterizes properties of the family of mappings $\{\Gamma_t; t \geq 0\}$ defined in (3.8). This is used in Section 5.2.2 to prove Lemma 3.16 on the solution of a transport equation.

5.2.1. Proof of the Properties of the Auxiliary Mapping .

PROOF OF LEMMA 3.6. We first prove property a. Fix $t \geq 0$. By Assumption I.a, the mapping $r \mapsto \overline{G}(r)\kappa(t)$ is continuously differentiable with derivative $-g(r)\kappa(t)$. Also, by Assumption I.c, g is continuously differentiable with derivative g' , and since $s \mapsto \kappa(s)$ is continuous, the mapping $r \mapsto \int_0^t \kappa(s)g(t+r-s)ds$ is continuously differentiable with derivative $\int_0^t \kappa(s)g'(t+r-s)ds$. Therefore, $\Gamma_t \kappa \in \mathbb{C}^1[0, \infty)$.

Furthermore, from (3.8), we have

$$(5.11) \quad |(\Gamma_t \kappa)(r)| \leq |\kappa(t)|\overline{G}(r) + \|\kappa\|_t \int_r^{t+r} g(s)ds \leq 2\|\kappa\|_t \overline{G}(r).$$

Since the right-hand side of (5.11) is a uniformly bounded and integrable function of $r \in (0, \infty)$, it follows that for each $t \geq 0$, $\Gamma_t \kappa \in \mathbb{L}^2(0, \infty)$. Furthermore, using Assumption I.c and (3.9), we have

$$(5.12) \quad |(\Gamma_t \kappa)'(r)| \leq |\kappa(t)|g(r) + \|\kappa\|_t \int_r^{t+r} |g'(s)|ds \leq \|\kappa\|_t \left(H\overline{G}(r) + H_2 \int_r^\infty \overline{G}(u)du \right).$$

Again, \overline{G} and $\int^\infty \overline{G}(u)du$ are bounded and integrable by Assumption II, and therefore, $(\Gamma_t \kappa)'$ also lies in $\mathbb{L}^2(0, \infty)$. Thus, $\Gamma_t \kappa \in \mathbb{H}^1(0, \infty)$. This completes the proof of part a.

For part b, let κ and $\tilde{\kappa}$ be functions in $\mathbb{C}[0, \infty)$. By linearity of the mapping Γ_t and the bounds (5.11) and (5.12), we have

$$(5.13) \quad \|\Gamma_t \kappa - \Gamma_t \tilde{\kappa}\|_{\mathbb{L}^2} \leq 2\|\overline{G}\|_{\mathbb{L}^2} \|\kappa - \tilde{\kappa}\|_t,$$

and

$$(5.14) \quad \|(\Gamma_t \kappa)' - (\Gamma_t \tilde{\kappa})'\|_{\mathbb{L}^2} \leq C_1 \|\kappa - \tilde{\kappa}\|_t,$$

with $C_1 \doteq H\|\overline{G}\|_{\mathbb{L}^2} + H_2\|\int^\infty \overline{G}(u)du\|_{\mathbb{L}^2}$, which is finite by Assumption I.c and Assumption II. The assertion in part b. then follows from (5.13) and (5.14).

For part **c**, fix $T \geq 0$. For every $0 \leq s < t \leq T$, by definition (3.8) of $\{\Gamma_t; t \geq 0\}$, Minkowski's integral inequality, Assumption **I.b** and Fubini's theorem, we have

$$\begin{aligned}
\|\Gamma_t \kappa - \Gamma_s \kappa\|_{\mathbb{L}^2} &\leq \|\overline{G}\|_{\mathbb{L}^2} |\kappa(t) - \kappa(s)| + \left\| \int_s^t \kappa(u) g(\cdot + t - u) du \right\|_{\mathbb{L}^2} \\
&\quad + \left\| \int_0^s \kappa(u) |g(\cdot + t - u) - g(\cdot + s - u)| du \right\|_{\mathbb{L}^2} \\
&\leq \|\overline{G}\|_{\mathbb{L}^2} |\kappa(t) - \kappa(s)| + \|\kappa\|_T H \left\| \int_0^{t-s} \overline{G}(\cdot + u) du \right\|_{\mathbb{L}^2} \\
&\quad + \|\kappa\|_T \left\| \int_0^s |g(\cdot + u) - g(\cdot + t - s + u)| du \right\|_{\mathbb{L}^2} \\
(5.15) \quad &\leq \|\overline{G}\|_{\mathbb{L}^2} |\kappa(t) - \kappa(s)| + \|\kappa\|_T H \|\overline{G}\|_{\mathbb{L}^2} |t - s| \\
&\quad + \|\kappa\|_T \int_0^T \|g(t - s + u + \cdot) - g(u + \cdot)\|_{\mathbb{L}^2} du.
\end{aligned}$$

The first two terms in (5.15) converge to zero as $|t - s| \rightarrow 0$ by the continuity of κ . Also, since Assumption **I** implies that g is square integrable, for every $u \in [0, T]$, the term $\|g(t - s + u + \cdot) - g(u + \cdot)\|_{\mathbb{L}^2}$ is bounded by $2\|g\|_{\mathbb{L}^2}$ and hence, the third term converges to zero as $t \rightarrow s$ by an application of the bounded convergence theorem and continuity of the translation map in the \mathbb{L}^2 norm.

Similarly, for every $0 \leq s < t \leq T$, by definition (3.9) of $(\Gamma \kappa)'$ and Assumption **I.c**,

$$\begin{aligned}
(5.16) \quad \|(\Gamma_t \kappa)' - (\Gamma_s \kappa)'\|_{\mathbb{L}^2} &\leq \|g\|_{\mathbb{L}^2} |\kappa(t) - \kappa(s)| + \|\kappa\|_T H_2 \|\overline{G}\|_{\mathbb{L}^2} |t - s| \\
&\quad + \|\kappa\|_T \int_0^T \|g'(\cdot + u) - g'(\cdot + t - s + u)\|_{\mathbb{L}^2} du.
\end{aligned}$$

Again, the first two terms on the right-hand side above converge to zero as $|t - s| \rightarrow 0$ by the continuity of κ , and the third term converges to zero as $|t - s| \rightarrow 0$ by continuity of the translation map in the \mathbb{L}^2 norm, boundedness of $\|g'\|_{\mathbb{L}^2}$ (see Assumption **I.c** and Remark 2.2) and the bounded convergence theorem. The $\mathbb{H}^1(0, \infty)$ -continuity of $t \mapsto \Gamma_t \kappa$ follows from (5.15) and (5.16). \square

5.2.2. Solution to the Transport Equation.

PROOF OF LEMMA 3.16. Define $\xi(t, r) = \Gamma_t F(r)$, for $t, r \geq 0$. First, we show that ξ is indeed a solution to (3.28). Since $F \in \mathbb{C}[0, \infty)$, by Lemma 3.6, $t \mapsto \xi(t, \cdot) = \Gamma_t F \in \mathbb{C}([0, \infty); \mathbb{H}^1(0, \infty))$ and for every $s \geq 0$, $\Gamma_s F$ has weak derivative

$$\partial_r \xi(t, r) = (\Gamma_s F)'(r) = -g(r)F(s) - \int_0^s F(u)g'(s + r - u)du, \quad r \in (0, \infty).$$

Because F , g and g' are continuous (see Assumption **I**), the mapping $(s, r) \mapsto (\Gamma_t F)'(r)$ is contin-

uous, and hence, locally integrable. Moreover, for $t, r \geq 0$,

$$\begin{aligned}
\int_0^t \partial_r \xi(s, r) ds + \overline{G}(r)F(t) &= \int_0^t (\Gamma_s F)'(r) ds + \overline{G}(r)F(t) \\
&= -g(r) \int_0^t F(s) ds - \int_0^t \int_0^s F(u) g'(r+s-u) du ds + \overline{G}(r)F(t) \\
&= -g(r) \int_0^t F(s) ds - \int_0^t F(u) (g(r+t-u) - g(r)) du + \overline{G}(r)F(t) \\
&= - \int_0^t F(u) g(r+t-u) du + \overline{G}(r)F(t) \\
&= \Gamma_t F(r),
\end{aligned}$$

where the application of Fubini's theorem in the second equality above is justified because g' and F are continuous and hence locally integrable. This shows that ξ satisfies (3.28).

Next, let $\tilde{\xi}$ be any function that satisfies properties 1 and 2 of Lemma 3.6 and equation (3.28). Then $\xi^\circ \doteq \xi - \tilde{\xi}$ also satisfies properties 1 and 2 of the lemma, as well as the following equation: for $t \geq 0$

$$(5.17) \quad \xi^\circ(t, r) = \int_0^t \partial_r \xi^\circ(s, r) ds, \quad \text{a.e. } r \in (0, \infty).$$

Fix $\delta > 0$ and $T > 0$, and for every $\epsilon \in (0, \delta)$, let ρ_ϵ be a regularizing kernel, that is,

$$\rho_\epsilon = \frac{1}{\epsilon} \rho\left(\frac{\cdot}{\epsilon}\right),$$

for a positive function $\rho \in \mathbb{C}_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \rho(x) dx = 1$ and $\text{supp}(\rho) \subseteq [-1, 1]$. For $t \geq 0$, define $\xi_\epsilon^\circ(t, \cdot) \doteq \xi^\circ(t, \cdot) * \rho_\epsilon$, and note that for every $\epsilon \in (0, \delta)$, $\xi_\epsilon^\circ(t, \cdot)$ is continuously differentiable on (ϵ, ∞) (in particular, $\xi_\epsilon^\circ(t, \cdot) \in \mathbb{C}^1[\delta, \infty)$), with $\partial_r \xi_\epsilon^\circ(t, \cdot) = (\partial_r \xi^\circ)(t, \cdot) * \rho_\epsilon$. Hence, $\partial_r [\xi_\epsilon^\circ(t, r)]^2 = 2\partial_r \xi_\epsilon^\circ(t, r) \xi_\epsilon^\circ(t, r)$, and since $\xi^\circ \in \mathbb{L}^2(0, \infty)$,

$$\lim_{r \rightarrow \infty} \xi_\epsilon^\circ(t, r) = \lim_{r \rightarrow \infty} \int_{r-\epsilon}^\infty \xi^\circ(t, x) \rho_\epsilon(r-x) dx \leq \lim_{r \rightarrow \infty} \|\xi^\circ(t, r-\epsilon+\cdot)\|_{\mathbb{L}^2} \|\rho_\epsilon\|_{\mathbb{L}^2} = 0.$$

In turn, this yields

$$(5.18) \quad (\xi_\epsilon^\circ)^2(t, r) = -2 \int_r^\infty \partial_r \xi_\epsilon^\circ(t, x) \xi_\epsilon^\circ(t, x) dx, \quad t, r \geq 0.$$

Moreover, convolving both sides of (5.17) with ρ_ϵ and using Fubini's theorem (which is justified since ρ_ϵ has a compact support), for every $\epsilon \in (0, \delta)$ we have

$$(5.19) \quad \xi_\epsilon^\circ(t, r) = \left(\int_0^t \partial_r \xi^\circ(s, \cdot) ds \right) * \rho_\epsilon(r) = \int_0^t (\partial_r \xi^\circ(s, \cdot) * \rho_\epsilon)(r) ds = \int_0^t \partial_r \xi_\epsilon^\circ(s, r) ds.$$

Also, for every fixed $r \geq \delta$ and every $t, s \geq 0$, we have

$$|\partial_r \xi_\epsilon^\circ(t, r) - \partial_r \xi_\epsilon^\circ(s, r)| \leq \int_0^\infty |\partial_r \xi^\circ(t, x) - \partial_r \xi^\circ(s, x)| \rho_\epsilon(r-x) dx \leq \|\rho_\epsilon\|_{\mathbb{L}^2} \|\partial_r \xi^\circ(t, \cdot) - \partial_r \xi^\circ(s, \cdot)\|_{\mathbb{L}^2}.$$

Since $\xi^\circ \in \mathbb{C}([0, \infty), \mathbb{H}^1(0, \infty))$ by the assumptions of the lemma, the right-hand side of the above display converges to zero as $s \rightarrow t$, and hence, the mapping $s \mapsto \partial_r \xi_\epsilon^\circ(s, r)$ is continuous for every $r \geq \delta$. Therefore, by equation (5.19) and the bounded convergence theorem, $\xi_\epsilon^\circ(\cdot, r) \in \mathbb{C}^1[0, \infty)$ for every $r \geq 0$, and the equation (5.19) can then be written as the classical (homogeneous) transport equation

$$(5.20) \quad \partial_t \xi_\epsilon^\circ(t, r) = \partial_r \xi_\epsilon^\circ(t, r), \quad t, r \geq 0,$$

with initial condition $\xi_\epsilon^\circ(0, r) \equiv 0$. In particular,

$$(5.21) \quad (\xi_\epsilon^\circ)^2(t, r) = 2 \int_0^t \partial_t \xi_\epsilon^\circ(s, r) \xi_\epsilon^\circ(s, r) ds, \quad t, r \geq 0.$$

Finally, applying equations (5.21), (5.20) and (5.18) in, respectively, the second, third and fifth equalities below, we have

$$(5.22) \quad \begin{aligned} \|\xi_\epsilon^\circ(t, \cdot)\|_{\mathbb{L}^2(\delta, \infty)}^2 &= \int_\delta^\infty (\xi_\epsilon^\circ)^2(t, r) dr \\ &= 2 \int_\delta^\infty \int_0^t \partial_t \xi_\epsilon^\circ(s, r) \xi_\epsilon^\circ(s, r) ds dr \\ &= 2 \int_\delta^\infty \int_0^t \partial_r \xi_\epsilon^\circ(s, r) \xi_\epsilon^\circ(s, r) ds dr \\ &= 2 \int_0^t \int_\delta^\infty \partial_r \xi_\epsilon^\circ(s, r) \xi_\epsilon^\circ(s, r) dr ds \\ &= - \int_0^t (\xi_\epsilon^\circ)^2(s, \delta) ds \leq 0, \end{aligned}$$

where the application of Fubini's theorem in the forth inequality above is justified by the bound

$$\int_0^t \int_\delta^\infty \partial_r \xi_\epsilon^\circ(s, r) \xi_\epsilon^\circ(s, r) dr ds \leq \int_0^t \|\partial_r \xi_\epsilon^\circ(s, \cdot)\|_{\mathbb{L}^2} \|\xi_\epsilon^\circ(s, \cdot)\|_{\mathbb{L}^2} ds < \infty.$$

which follows from the the assumption that $\xi^\circ \in \mathbb{C}([0, \infty), \mathbb{H}^1(0, \infty))$. The inequality (5.22) implies $\xi_\epsilon^\circ \equiv 0$ on $[0, \infty) \times [\delta, \infty)$ for every $\epsilon < \delta$. Since for every $t \geq 0$, $\xi_\epsilon^\circ(t, \cdot) \rightarrow \xi^\circ(t, \cdot)$ in $\mathbb{L}_{\text{loc}}^2(0, \infty)$ (see, e.g., [15, Appendix C.4, Theorem 6.(iv)]), we conclude that $\xi(t, \cdot) \equiv 0$ for every $t \geq 0$, which completes the proof. \square

APPENDIX A: PROPERTIES OF THE RENEWAL EQUATION

PROOF OF PROPOSITION 4.4.. For part a, let $U = \sum_{n=0}^\infty G^{*n}$ be the renewal function associated with the distribution function G , where G^{*n} denotes the n -fold convolution of G . Since the service distribution with cdf G has probability density function g , by [4, Proposition 2.7, Section V], U has density $u = U * g$, which satisfies the equation $u = g + g * u$. Moreover, since g is continuous (and hence locally bounded), u is also locally bounded. Define the function $\varphi_* \doteq f + u * f$. Then the local integrability of f and local boundedness of u imply the local integrability of φ_* , and, using the distributive and associative properties of the convolution operation, we have

$$g * \varphi_* = g * (f + u * f) = g * f + (g * u) * f = g * f + (u - g) * f = u * f = \varphi_* - f.$$

Therefore, φ_* is a solution of the equation $\varphi = f + g * \varphi$ given in (4.12).

To show that φ_* is the unique solution to (4.12) that lies in $\mathbb{L}_{\text{loc}}^1(0, \infty)$, let $\varphi_i \in \mathbb{L}_{\text{loc}}^1(0, \infty)$, $i = 1, 2$, be two solutions to (4.12). Then $\varphi = \varphi_1 - \varphi_2$ is a solution to the equation $\varphi = g * \varphi$. For $\varepsilon > 0$, let $\eta_\varepsilon(x) \doteq \mathbb{1}_{(0, \varepsilon)}(x)/\varepsilon$, and note that the function $\varphi_\varepsilon \doteq \varphi * \eta_\varepsilon$ satisfies

$$\varphi_\varepsilon = \varphi * \eta_\varepsilon = g * \varphi * \eta_\varepsilon = g * \varphi_\varepsilon.$$

Also, for every $T < \infty$,

$$|\varphi_\varepsilon(t)| \leq \frac{1}{\varepsilon} \int_0^T |\varphi(x)| dx < \infty, \quad t \in [0, T],$$

where the finiteness holds since $\varphi \in \mathbb{L}_{\text{loc}}^1(0, \infty)$. Hence, φ_ε is locally bounded and satisfies the renewal equation $\varphi_\varepsilon = g * \varphi_\varepsilon$ (i.e., with “input function” identically equal to zero). However, by [4, Theorem 2.4, p. 146], this implies $\varphi_\varepsilon \equiv 0$. Since this holds for every $\varepsilon > 0$, this implies $\varphi \equiv 0$.

To see why part b holds, first note that $f \in \mathbb{L}^2(0, \infty)$ implies $f \in \mathbb{L}_{\text{loc}}^1(0, \infty)$. Therefore, $\varphi_* = f + u * f$ is a solution of (4.12) from part a, and, moreover, it satisfies

$$|\varphi_*(t)| \leq |f(t)| + |u * f(t)| \leq |f(t)| + \left| \int_0^t f(t-s)(u(s)-1)ds \right| + \left| \int_0^t f(s)ds \right|,$$

and hence, recalling the notation $\mathcal{I}_f(t) = \int_0^t f(s)ds$,

$$(A.1) \quad \|\varphi_*\|_{\mathbb{L}^2} \leq \|f\|_{\mathbb{L}^2} + \|f * (u-1)\|_{\mathbb{L}^2} + \|\mathcal{I}_f\|_{\mathbb{L}^2}.$$

With some abuse of notation, we also let U denote the renewal measure associated with the distribution G , and let l^+ denote Lebesgue measure on $(0, \infty)$. Then, since $u-1$ is the density of the signed measure $U - l^+$ with respect to l^+ on $(0, \infty)$, we have

$$\int_0^\infty |u(s)-1|ds = \|U - l^+\|_{TV},$$

where $\|\cdot\|_{TV}$ is the total variation norm. Since G has a finite second moment (and has mean 1), it follows from [34, eqn. (6.10), p. 86] (with $\lambda = 1$) that $\|U - l^+\|_{TV}$ is finite and hence, $u-1 \in \mathbb{L}^1(0, \infty)$. By Young’s inequality, we then have $\|f * (u-1)\|_{\mathbb{L}^2} \leq \|u-1\|_{\mathbb{L}^1} \|f\|_{\mathbb{L}^2}$. Substituting this inequality into (A.1), we obtain the bound (4.13) with $c_1 = 1 + \|u-1\|_{\mathbb{L}^1} < \infty$ and $c_2 = 1$.

Finally, to see why c holds, note that since $f \in \mathbb{C}[0, \infty)$ and u is bounded on finite intervals, $u * f$ also lies in $\mathbb{C}[0, \infty)$, and hence, so does $\varphi_* = f + u * f$. Moreover, for every $f^1, f^2 \in \mathbb{C}[0, \infty)$ and corresponding solutions φ_*^1, φ_*^2 , we have

$$\|\varphi_*^1 - \varphi_*^2\|_T \leq (1 + U(T)) \|f^1 - f^2\|_T, \quad \forall T \geq 0,$$

and the continuity claim follows. \square

APPENDIX B: PROPERTIES OF $\mathbb{H}^1(0, \infty)$

PROOF OF LEMMA 2.3.c. The first claim follows from the following elementary inequality: for every $f_1, f_2 \in \mathbb{H}^1(0, \infty)$,

$$\|f_1(t+\cdot) - f_2(t+\cdot)\|_{\mathbb{H}^1}^2 = \int_t^\infty (f_1(u) - f_2(u))^2 du + \int_t^\infty (f_1'(u) - f_2'(u))^2 du \leq \|f_1 - f_2\|_{\mathbb{H}^1}.$$

For the second claim, fix $f \in \mathbb{L}^2(0, \infty)$ and $\epsilon > 0$. Since $\mathbb{C}_c(0, \infty)$ is dense in $\mathbb{L}^2(0, \infty)$, there exists a function \tilde{f} that is uniformly continuous on $(0, \infty)$ such that $\|f - \tilde{f}\|_{\mathbb{L}^2} \leq \epsilon$. Therefore, for every $t, t_0 \in (0, \infty)$,

$$\begin{aligned} \|f(t + \cdot) - f(t_0 + \cdot)\|_{\mathbb{L}^2} &\leq \|f(t + \cdot) - \tilde{f}(t + \cdot)\|_{\mathbb{L}^2} + \|\tilde{f}(t + \cdot) - \tilde{f}(t_0 + \cdot)\|_{\mathbb{L}^2} + \|\tilde{f}(t_0 + \cdot) - f(t_0 + \cdot)\|_{\mathbb{L}^2} \\ &\leq 2\epsilon + \|\tilde{f}(t + \cdot) - \tilde{f}(t_0 + \cdot)\|_{\mathbb{L}^2}. \end{aligned}$$

Taking the limit as $t \rightarrow t_0$ on both sides of the last inequality, by the uniform continuity of \tilde{f} and the dominated convergence theorem, we have

$$\lim_{t \rightarrow t_0} \|f(t + \cdot) - f(t_0 + \cdot)\|_{\mathbb{L}^2} \leq 2\epsilon + \lim_{t \rightarrow t_0} \|\tilde{f}(t + \cdot) - \tilde{f}(t_0 + \cdot)\|_{\mathbb{L}^2} = 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this shows that the translation map $t \mapsto f(t + \cdot)$ is continuous in $\mathbb{L}^2(0, \infty)$. If $f \in \mathbb{H}^1[0, \infty)$, then $f' \in \mathbb{L}^2(0, \infty)$ and so the above argument also shows that the map $t \mapsto f'(t + \cdot)$ is continuous in $\mathbb{L}^2(0, \infty)$, which proves the continuity of $t \mapsto f(t + \cdot)$ in $\mathbb{H}^1(0, \infty)$. Finally, by definition,

$$\|f(t + \cdot)\|_{\mathbb{L}^2}^2 = \int_0^\infty f^2(t + x) dx = \int_t^\infty f^2(x) dx.$$

Since $f \in \mathbb{L}^2(0, \infty)$, the right-hand side above converges to zero as $t \rightarrow \infty$. Similarly, since $f' \in \mathbb{L}^2$, $\lim_{t \rightarrow \infty} \|f'(t + \cdot)\|_{\mathbb{L}^2} = 0$, and (2.16) follows. \square

APPENDIX C: A CONTINUOUS VERSION OF THE ASYMPTOTIC COUPLING THEOREM

In this section, for completeness, we prove the continuous version of Corollary 2.2 of [22], as stated in Proposition 4.1. The proof is a straightforward adaptation of the proof of [22, Theorem 1.1 and Corollary 2.2] to the continuous time setting (see also [14, Theorem 2] and [37, Lemma 8.5], where a continuous version is used). In what follows, recall that for any semi-group of measurable operators $\phi^s : (\mathcal{X}^{\mathbb{R}_+}, \mathcal{B}(\mathcal{X})^{\mathbb{R}_+}) \mapsto (\mathcal{X}^{\mathbb{R}_+}, \mathcal{B}(\mathcal{X})^{\mathbb{R}_+})$, $s \geq 0$, on the probability space $(\mathcal{X}^{\mathbb{R}_+}, \mathcal{B}(\mathcal{X})^{\mathbb{R}_+})$, a probability measure μ on $(\mathcal{X}^{\mathbb{R}_+}, \mathcal{B}(\mathcal{X})^{\mathbb{R}_+})$ is said to be an invariant measure of $\{\phi^s\}$ if $\phi_\#^s \mu = \mu$ for all $s \geq 0$. Also, a set $A \in \mathcal{B}(\mathcal{X})^{\mathbb{R}_+}$ is said to be an invariant set of $\{\phi^s\}$ if $(\phi^s)^{-1}(A) = A$ for all $s \geq 0$. Finally, an invariant measure μ of $\{\phi^s\}$ is said to be ergodic if $\mu(A) \in \{0, 1\}$ for every invariant set A of $\{\phi^s\}$.

PROOF OF PROPOSITION 4.1. Define the family $\{\Theta_s; s \geq 0\}$ of shift operators Θ_s as $\Theta_s x(\cdot) \doteq x(s + \cdot)$, $x \in \mathcal{X}^{\mathbb{R}_+}$, and note that it forms a semigroup of measurable operators from $(\mathcal{X}^{\mathbb{R}_+}, \mathcal{B}(\mathcal{X})^{\mathbb{R}_+})$ to itself. Let μ_1 and μ_2 be two invariant distributions of $\{\mathcal{P}_t\}$. Given the ergodic decomposition of invariant measures, we can assume without loss of generality that μ_1 and μ_2 are both ergodic invariant distributions of $\{\mathcal{P}_t\}$. Hence, defining $m_i \doteq P^{\mu_i}$, $i = 1, 2$, to be the distribution of Markov processes with transition semigroup $\{\mathcal{P}_t\}$ and initial condition μ_i , m_1 and m_2 are ergodic invariant distributions for the semigroup $\{\Theta_s\}$.

First, we extend the definition of the measurable map Υ in the statement of the proposition from $A \times A$ to the whole space $\mathcal{X} \times \mathcal{X}$ by setting $\Upsilon_{y, \tilde{y}} = P^y \times P^{\tilde{y}}$ for $(y, \tilde{y}) \notin A \times A$. Next, let $\bar{\Upsilon} \in \mathbb{M}_1(\mathcal{X}^{\mathbb{R}_+} \times \mathcal{X}^{\mathbb{R}_+})$ be the measure given by

$$\bar{\Upsilon}(B) \doteq \int_{\mathcal{X} \times \mathcal{X}} \Upsilon_{y, \tilde{y}}(B) \mu_1(dy) \mu_2(d\tilde{y}), \quad B \in \mathcal{B}(\mathcal{X})^{\mathbb{R}_+} \otimes \mathcal{B}(\mathcal{X})^{\mathbb{R}_+}.$$

Note that by construction, $\tilde{\Upsilon} \in \tilde{\mathcal{C}}(m_1, m_2)$. Also, since by conditions (I) and (III) of the proposition, $\tilde{\Upsilon}_{y, \tilde{y}}(\mathcal{D}) > 0$ for all $(y, \tilde{y}) \in A \times A$ and $\mu_1(A), \mu_2(A) > 0$, it follows that $\tilde{\Upsilon}(\mathcal{D}) > 0$.

Now, fix any bounded, Lipschitz function ϕ on \mathcal{X} , and define $\tilde{\phi} : \mathcal{X}^{\mathbb{R}_+} \mapsto \mathbb{R}$ by $\tilde{\phi}(x) \doteq \phi(x(0))$, $x \in \mathcal{X}^{\mathbb{R}_+}$. By (the continuous-time version of) Birkhoff's ergodic theorem (see, e.g., [31, Theorem 1 of Section 1.2]), for $i = 1, 2$, there exist sets $B_i^\phi \in \mathcal{B}(\mathcal{X})^{\mathbb{R}_+}$ with $m_i(B_i^\phi) = 1$ such that for every $x_i \in B_i^\phi$,

$$(C.1) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \phi(x_i(s)) ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \tilde{\phi}(\Theta_s x_i) ds = \int_{\mathcal{X}^{\mathbb{R}_+}} \tilde{\phi}(x_i) m_i(dx) = \int_{\mathcal{X}} \phi(z) \mu_i(dz).$$

Now, since $\tilde{\Upsilon} \in \tilde{\mathcal{C}}(m_1, m_2)$, for $i = 1, 2$, the marginal $\Pi_{\#}^{(i)} \tilde{\Upsilon}$ is absolutely continuous with respect to m_i , which in turn implies $\Pi_{\#}^{(i)} \tilde{\Upsilon}(B_i^\phi) = 1$ because $m_i(B_i^\phi) = 1$. Thus, we have $\tilde{\Upsilon}(B_1^\phi \times B_2^\phi) = 1$. Also, since $\tilde{\Upsilon}(\mathcal{D}) > 0$, defining $\bar{\mathcal{D}} \doteq \mathcal{D} \cap (B_1^\phi \times B_2^\phi)$, we have $\tilde{\Upsilon}(\bar{\mathcal{D}}) > 0$, and in particular, $\bar{\mathcal{D}}$ is not empty. Take any $(x_1, x_2) \in \bar{\mathcal{D}}$. Then $(x_1, x_2) \in \mathcal{D}$ and by (C.1), we conclude that

$$\begin{aligned} \left| \int_{\mathcal{X}} \phi(z) \mu_1(dz) - \int_{\mathcal{X}} \phi(z) \mu_2(dz) \right| &= \lim_{t \rightarrow \infty} \frac{1}{t} \left| \int_0^t (\phi(x_1(s)) - \phi(x_2(s))) ds \right| \\ &\leq \lim_{t \rightarrow \infty} \frac{C_\phi}{t} \int_0^t d((x_1(s)), x_2(s)) ds \\ &= 0, \end{aligned}$$

where C_ϕ is the Lipschitz constant of ϕ and the last equality follows from the definition (4.2) of \mathcal{D} . Therefore, $\int_{\mathcal{X}} \phi(z) \mu_1(dz) = \int_{\mathcal{X}} \phi(z) \mu_2(dz)$ for every bounded Lipschitz function ϕ , and hence, $\mu_1 = \mu_2$. \square

APPENDIX D: VERIFICATION OF ASSUMPTIONS FOR CERTAIN FAMILIES OF DISTRIBUTIONS

In this section, we show that a large class of distributions of interest satisfy our assumptions.

LEMMA D.1. *Assumption I and Assumption II are satisfied when G belongs to one of the following families of distributions:*

1. *Generalized Pareto distributions with location parameter $\mu = 0$ (a.k.a. Lomax distribution), and shape parameter $\alpha > 2$.*
2. *The log-normal distribution with location parameter $\mu \in (-\infty, \infty)$ and scale parameter $\sigma > 0$.*
3. *The Gamma distribution with shape parameter $\alpha \geq 2$.*
4. *Phase-type distributions.*

PROOF. *Family 1.* The Lomax distribution (equivalently, the generalized Pareto distribution with location parameter $\mu = 0$) with scale parameter $\lambda > 0$ and shape parameter $\alpha > 0$ has the following complementary c.d.f.:

$$\bar{G}(x) = \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \quad x \geq 0.$$

Elementary calculations show that for $\alpha > 1$, the distribution has a finite mean, which is equal to $\lambda/(\alpha - 1)$. In particular, the distribution has mean 1 when $\lambda = \alpha - 1$. The probability density

function (p.d.f.) g of G clearly exists, is continuously differentiable and satisfies

$$g(x) = \frac{\alpha}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)}, \quad \text{and} \quad g'(x) = -\frac{\alpha(\alpha+1)}{\lambda^2} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha+2)},$$

for $x \in (0, \infty)$. Thus, the hazard rate function h is equal to

$$h(x) = \frac{g(x)}{\overline{G}(x)} = \frac{\alpha}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-1}, \quad x \geq 0,$$

which is uniformly bounded by α/λ , and

$$h_2(x) = \frac{g'(x)}{\overline{G}(x)} = -\frac{\alpha(\alpha+1)}{\lambda^2} \left(1 + \frac{x}{\lambda}\right)^{-2}, \quad x \geq 0,$$

which shows that $|h_2|$ is uniformly bounded by $\alpha(\alpha+1)/\lambda^2$. Therefore, Assumption I is satisfied. Moreover, as $x \rightarrow \infty$,

$$\overline{G}(x) = x^{-\alpha} \left(\frac{1}{x} + \frac{1}{\lambda}\right)^{-\alpha} = \mathcal{O}(x^{-\alpha}),$$

and hence, Assumption II holds when $\alpha > 2$.

Family 2. The complementary c.d.f. of the log-normal distribution with location parameter $\mu \in \mathbb{R}$ and shape parameter $\sigma > 0$ has the form

$$\overline{G}_{\mu,\sigma}(x) = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{\log x - \mu}{\sqrt{2}\sigma}\right), \quad x \geq 0,$$

where $\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} dt$ is the error function. Simple calculations show that the mean is given by $e^{\mu+\sigma^2/2}$, which is equal to 1 when $\mu = -\sigma^2/2$. Fix $\sigma > 0$. For any $\mu \in \mathbb{R}$, $\overline{G}_{\mu,\sigma}(x) = \overline{G}_{0,\sigma}(cx)$, with $c \doteq e^{-\mu}$, for all $x \geq 0$. Therefore, it suffices to verify Assumption I.b, Assumption I.c and Assumption II for $\overline{G} \doteq \overline{G}_{0,\sigma}$. The p.d.f. g of $G = G_{0,\sigma}$ exists, is continuous, and is given explicitly by

$$g(x) = \frac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{(\log x)^2}{2\sigma^2}} = \frac{1}{x\sigma} \phi\left(\frac{\log x}{\sigma}\right), \quad x > 0,$$

where ϕ is the p.d.f. of the standard Gaussian distribution. The p.d.f. g itself is continuously differentiable with derivative

$$g'(x) = -\frac{\log x + \sigma^2}{x^2\sigma^3} \phi\left(\frac{\log x}{\sigma}\right), \quad x > 0.$$

The complementary c.d.f. \overline{G} can be written as

$$\overline{G}(x) = Q\left(\frac{\log x}{\sigma}\right), \quad x \geq 0,$$

where Q is the function $Q(z) = 1/2 + 1/2 \operatorname{erf}(z/\sqrt{2})$, which satisfies the bounds [7, equation (8)]

$$(D.1) \quad Q(z) \geq \frac{z}{1+z^2} \phi(z), \quad z \geq 0.$$

For $x \geq 0$, set $z_x \doteq \log(x)/\sigma$. Then, using the bound (D.1), for $x \geq e^\sigma$ (and hence $z_x \geq 1$) we have

$$h(x) = \frac{\phi(z_x)}{x\sigma Q(z_x)} \leq \frac{(1+z_x^2)}{\sigma z_x e^{\sigma z_x}} \leq \frac{(1+z_x^2)}{\sigma} e^{-\sigma z_x}.$$

Moreover, for $x \in [e^\sigma, \infty)$, since $\log x < x$,

$$\frac{g'(x)}{g(x)} = \frac{\log x + \sigma^2}{x\sigma^2} \leq \frac{1}{\sigma^2} + e^{-\sigma},$$

and hence, $h_2 = (g'/g)h$ is also bounded on $[e^\sigma, \infty)$. Moreover,

$$\lim_{x \rightarrow 0} g(x) = \lim_{z \rightarrow -\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\sigma z - \frac{z^2}{2}} = 0,$$

and

$$\lim_{x \rightarrow 0} g'(x) = \lim_{z \rightarrow -\infty} -\frac{z + \sigma}{\sqrt{2\pi}\sigma^2} e^{-\sigma z - \frac{z^2}{2}} = 0.$$

Since g and g' are continuous and \overline{G} is decreasing, it follows from the last two displays that h and h_2 are also bounded on $(0, e^\sigma)$. Therefore, Assumptions I holds.

Moreover, it is straightforward to see that for a random variable X with log-normal distribution,

$$\mathbb{E}[X^n] = e^{n\mu + \frac{n^2\sigma^2}{2}} < \infty.$$

In other words, all moments of the log-normal distribution exist, and in particular, Assumption II holds.

Family 3. The complementary c.d.f. of a Gamma distribution with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$ is given by

$$\overline{G}(x) = \frac{1}{\Gamma(\alpha)} \Gamma(\alpha, \beta x), \quad x > 0,$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete Gamma function. The mean is α/β , which is equal to one when $\alpha = \beta$. The p.d.f. g is equal to

$$g(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0,$$

which is itself continuously differentiable on $(0, \infty)$, with derivative

$$g'(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} (\alpha - 1 - \beta x) x^{\alpha-2} e^{-\beta x}, \quad x > 0.$$

When $\alpha \geq 2$,

$$\lim_{x \rightarrow 0} h(x) = \lim_{x \rightarrow 0} g(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lim_{x \rightarrow 0} x^{\alpha-1} = 0.$$

Also, by L'Hôpital's rule,

$$\lim_{x \rightarrow \infty} h(x) = \lim_{x \rightarrow \infty} \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\int_{\beta x}^{\infty} t^{\alpha-1} e^{-t} dt} = \lim_{x \rightarrow \infty} \beta^\alpha \frac{(\alpha-1)x^{\alpha-2} e^{-\beta x} - \beta x^{\alpha-1} e^{-\beta x}}{-\beta^\alpha x^{\alpha-1} e^{-\beta x}} = \beta.$$

Moreover, again when $\alpha \geq 2$,

$$\lim_{x \rightarrow 0} h_2(x) = \lim_{x \rightarrow 0} g'(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} (\alpha - 1) \lim_{x \rightarrow 0} x^{\alpha-2} < \infty$$

and by L'Hôpital's rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} h_2(x) &= \lim_{x \rightarrow \infty} \frac{\beta^\alpha (\alpha - 1 - \beta x) x^{\alpha-2} e^{-\beta x}}{\int_{\beta x}^{\infty} t^{\alpha-1} e^{-t} dt} \\ &= \lim_{x \rightarrow \infty} \frac{\beta^\alpha (\beta^2 x^{\alpha-1} - 2\beta(\alpha - 1)x^{\alpha-2} + (\alpha - 1)(\alpha - 2)x^{\alpha-3}) e^{-\beta x}}{-\beta^\alpha x^{\alpha-1} e^{-\beta x}} \\ &= -\beta^2. \end{aligned}$$

Since h and h_2 are continuous on $(0, \infty)$, the last four displays show that Assumption I holds for $\alpha \geq 2$.

Moreover, the Gamma distribution has finite exponential moments in a neighborhood of the origin, and in particular, Assumption II holds.

Family 4. A phase-type distribution with size m , an $m \times m$ -subgenerator matrix \mathbf{S} (which has eigenvalues with negative real part) and probability row vector $\boldsymbol{\alpha}$, the complementary c.d.f. function has the representation

$$\overline{G}(x) = \sum_{j=1}^m p_j(x) = \boldsymbol{\alpha} e^{x\mathbf{S}} \mathbf{1}, \quad x \geq 0,$$

where $\mathbf{1}$ is an $m \times 1$ column vector of ones and

$$\mathbf{p}(x) = [p_1(x), \dots, p_m(x)] \doteq \boldsymbol{\alpha} e^{x\mathbf{S}}, \quad x \geq 0.$$

Note that the vector $\boldsymbol{\alpha}$ can be chosen such that the mean $-\boldsymbol{\alpha} \mathbf{S}^{-1} \mathbf{1}$ is set to one. Defining $\boldsymbol{\mu} = [\mu_1, \dots, \mu_m] \doteq -\mathbf{S} \mathbf{1}$, the probability density function g can be written as

$$g(x) = -\boldsymbol{\alpha} e^{x\mathbf{S}} \mathbf{S} \mathbf{1} = \boldsymbol{\alpha} e^{x\mathbf{S}} \boldsymbol{\mu} = \sum_{j=1}^m p_j(x) \mu_j,$$

and is continuous on $(0, \infty)$. Therefore, Assumption I.a holds. The p.d.f. g is continuously differentiable with derivative

$$g'(x) = -\boldsymbol{\alpha} e^{x\mathbf{S}} \mathbf{S}^2 \mathbf{1} = \boldsymbol{\alpha} e^{x\mathbf{S}} \boldsymbol{\nu}, \quad x > 0,$$

where $\boldsymbol{\nu} = [\nu_1, \dots, \nu_m]^T \doteq -\mathbf{S}^2 \mathbf{1}$. The hazard rate function h satisfies

$$h(x) = \frac{\sum_{j=1}^m p_j(x) \mu_j}{\sum_{j=1}^m p_j(x)} \leq \max_{j=1, \dots, m} \mu_j < \infty, \quad x \geq 0.$$

Hence, h is uniformly bounded on $[0, \infty)$. Moreover,

$$|h_2(x)| = \frac{\left| \sum_{j=1}^m p_j(x) \nu_j \right|}{\sum_{j=1}^m p_j(x)} \leq \max_{j=1, \dots, m} |\nu_j| < \infty, \quad x > 0.$$

Therefore, Assumptions I.b and I.c hold.

Moreover, for a random variable X with a phase-type distribution,

$$\mathbb{E}[X^n] = (-1)^n n! \boldsymbol{\alpha} \mathbf{S}^{-n} \mathbf{1} < \infty, \quad n \in \mathbb{N}.$$

Therefore, all moments are finite and in particular, Assumption II holds. □

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